Christina Sormani - Sliced Filling Volume and Intrinsic Flat Convergence

Consider a sequence of Manifolds M_j. When does a subsequence converge?
How singular is the limit space

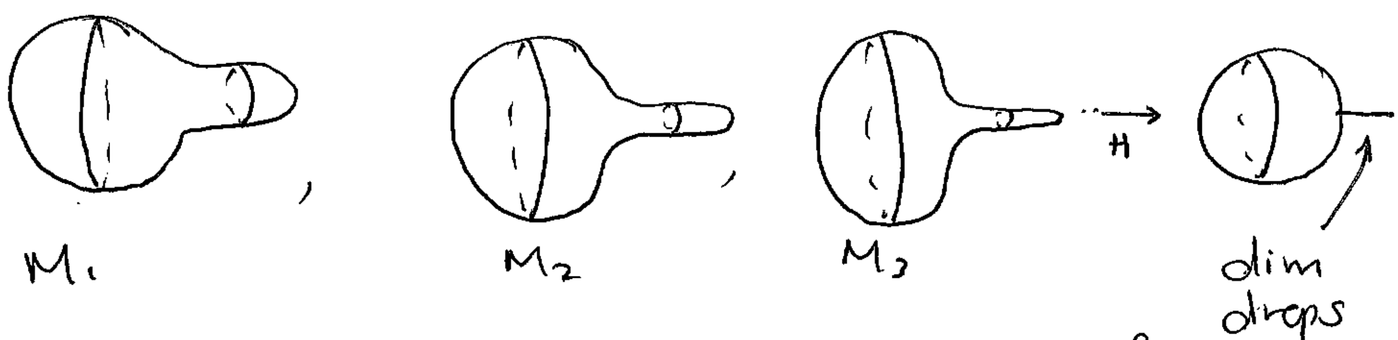
Outline:
* Review of Intrinsic Flat Convergence
* Tetrahedral Compactness (joint w/ Portegies)

Convergence of Submanifolds in $\mathbb{E}^N$

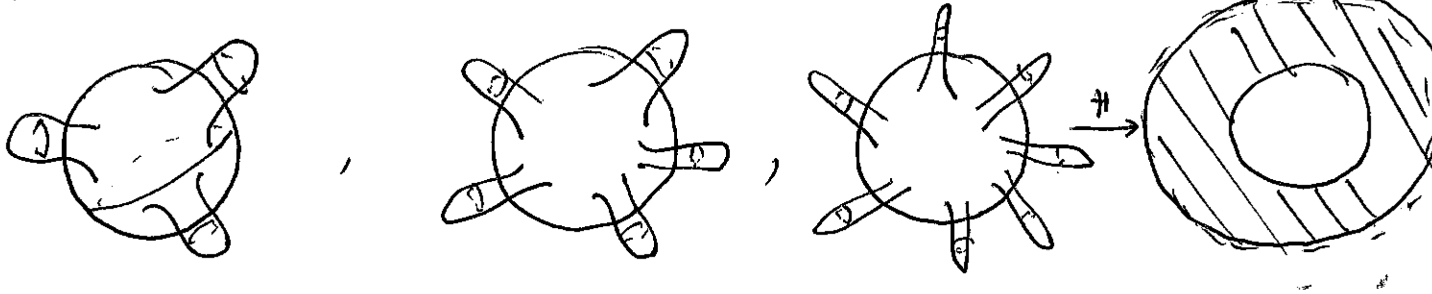
$M_j^m \subset \mathbb{E}^N$, Hausdorff distance:

$$d_H^{\mathbb{E}^N}(M_1^m, M_2^m) = \inf\{ r : M_1^m \subset T_r(M_2^m) \text{ and } M_2^m \subset T_r(M_1^m) \}$$

T_r ← tubular nbhd of length r



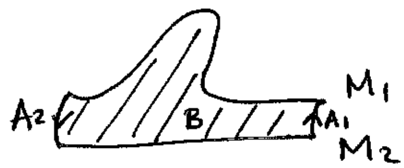
Hausdorff limits have no regularity or dimension for the limits (in our example, dim was diff for diff parts)



~~Eleudor~~ Federer - Fleming introduced Flat distance to get regularity + dimension for limits

weighted volume

$$d_F^{\mathbb{E}^N}(M_1^m, M_2^m) = \inf\{ \mathbb{M}(A^m) + \mathbb{M}(B^{m+1}) : [|M_1|] - [|M_2|] = A + \partial B \}$$



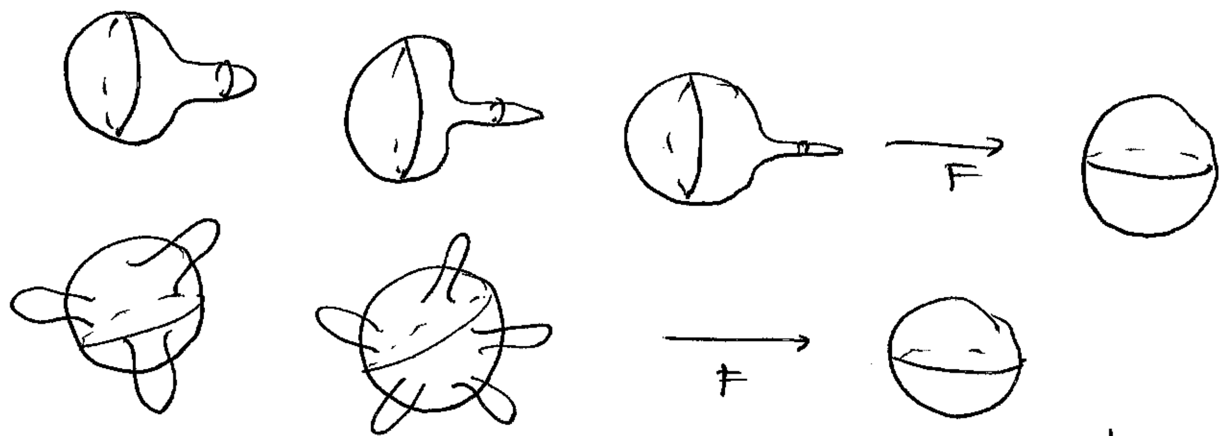
What is nice is:

COMPACTNESS THM

$M_j^m \subseteq K \subseteq \mathbb{E}^N$ K compact

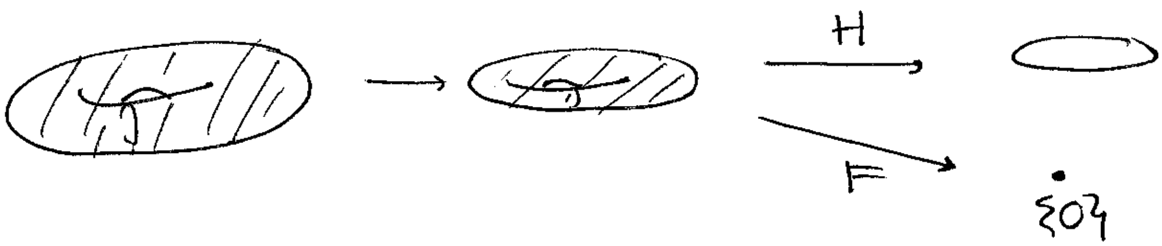
$\mathbb{M}(M_j^m) \leq V_0, \quad \mathbb{M}(\partial M_j^m) \leq A$

subseq $M_j \xrightarrow{F} M_\infty$ cantably H^m rectifiable.



F limit has the same dim as the elements in the sequence.

Trouble w/ Flat limits



collapse

$$d_F(M_i^m, 0) = \inf\{\mathbb{M}(A) + \mathbb{M}(B) : [\![M_i]\!] = A + \partial B\}$$

Flat limits also dissappear under cancellation

area is going to zero

You can extend this to a metric space Z instead of $\mathbb{E}^N$. Easy to do with Hausdorff

Ambrosio-Kircheinn Wegner (this was very hard) extend flat distance on Z.

$\longrightarrow$ cont.

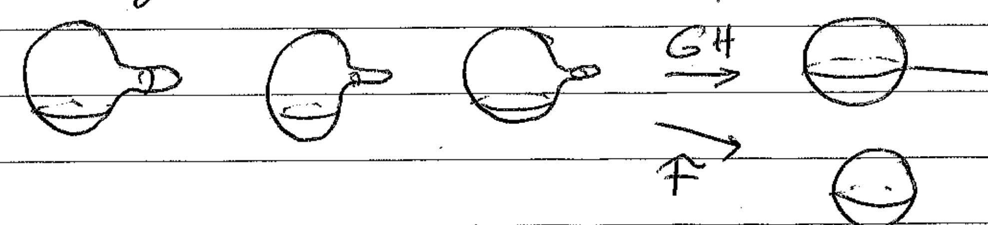
Convergence of Riemannian Manifolds

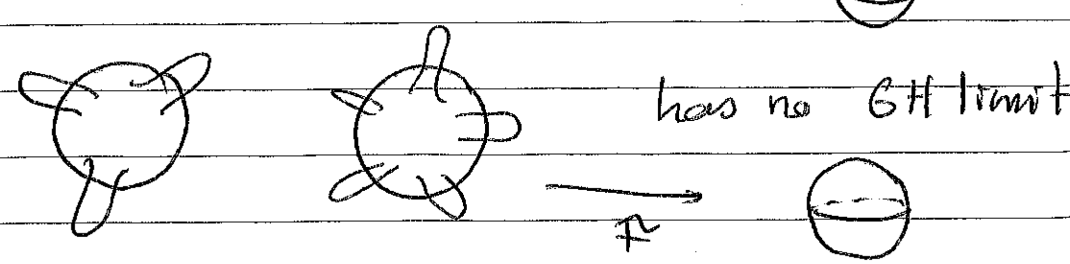
Gromov: $d_{GH}(M_1, M_2) = \inf\{ d_H^Z(\phi_1(M_1), \phi_2(M_2)) : \phi_i : M_i \to Z$ are distance preserving metric spaces $Z \}$

Distance preserving
$$d_Z(\phi_1(p), \phi_1(q)) = d_{M_1}(p, q)$$

Intrinsic Flat distance (Sormani-Wenger)
Oriented Riemannian manifolds
$$d_{\mathcal{F}}(M_1^m, M_2^m) = \inf\{ d_{\mathcal{F}}^Z(\phi_{1\#}[[M_1]], \phi_{2\#}[[M_2]]) : \text{dist preserving } \phi_i : M_i \to Z,\ Z \text{ complete metric spaces} \}$$

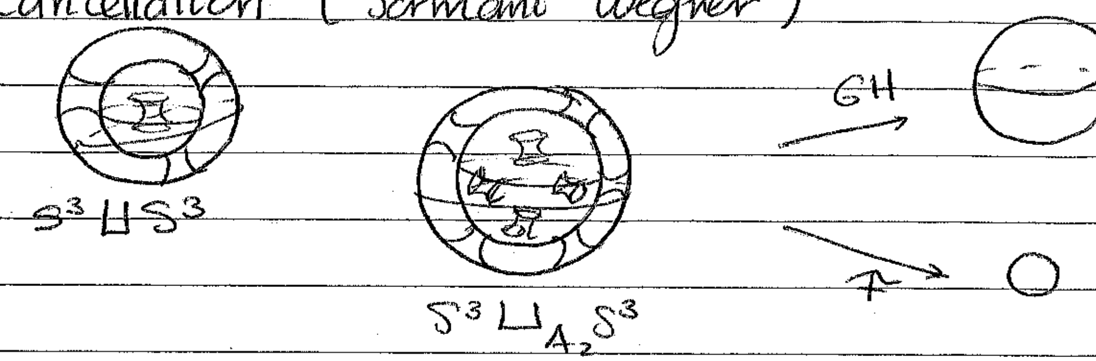
Limits are countably $\mathcal{H}^m$ rectifiable metric spaces with $\mathbb{Z}$ weight and orientation possibly 0 space



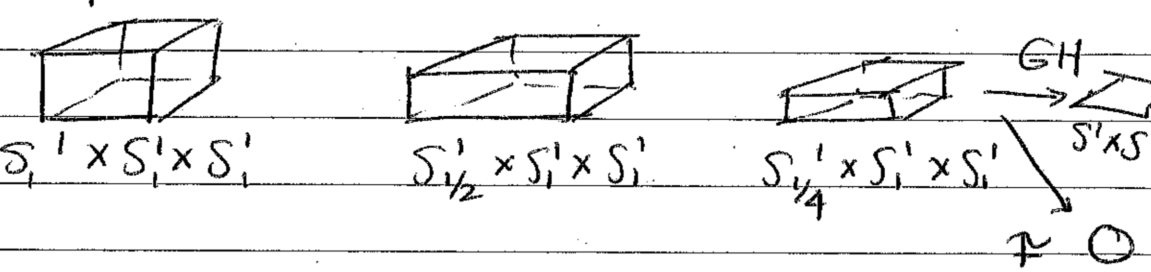


Can there be collapsing or cancelling for the intrinsic flat limit

Cancellation (Sormani Wegner)



Collapse



Gromov's Compactness thm
If Riemann M_j^m have uniform diameter $(M_j) \le D_0$ and # of disjoint balls of radius $r \le N(r)$ uniform, then a subseq $M_j^m \xrightarrow{GH} M_\infty$ (compact metric space)

Wenger's Compactness theorem

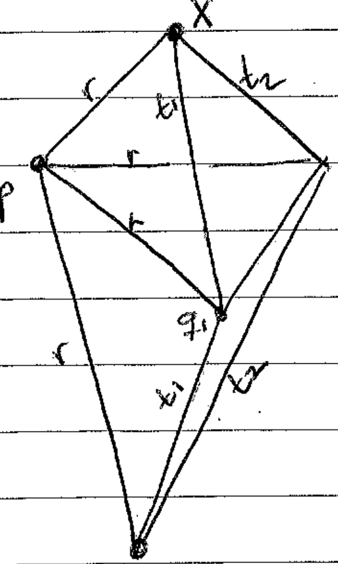
If M_j^m oriented $\mathrm{vol}(M_j^m) \le V_0$
$\mathrm{vol}(\partial M_j^m) \le A_0$ $\mathrm{diam}(M_j^m) \le D_0$
then a subsequence $M_j^m \xrightarrow{\mathcal{F}} M_\infty^m$
cantably H^m rectifiable but possibly $M_\infty^m = 0$

Lang-Wenger have a pointed version

Tetrahedral Compactness Theorem (Portegies Sormani)

$\partial M_j = \emptyset$

Given $r_0 > 0$ $\beta \in (0,1)$ $C > 0$ $V_0 > 0$
Suppose M_j^3 satisfy the C-β tetrahedral property $\forall$ balls $B_p(r) \subseteq M_j^3$ $r \le r_0$

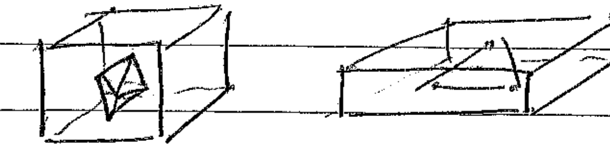


X and Y exist

X and Y are not close $d(X,Y) \geq Cr$

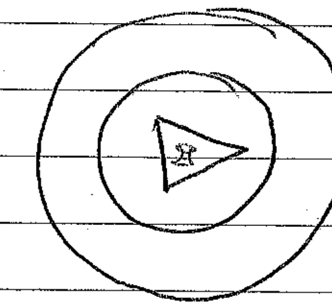
$\exists$ q_1 $q_2 \in \partial B(p,r)$
s.t. $t_1, t_2 \in [(1-\beta)r, (1+\beta)r]$
we have
$\inf\{d(x,y) : x \neq y,$
$x, y \in \partial B(p,q) \cap \partial B(q_1,t_1)$
$\cap \partial B(q_2,t_2)\} \in [Cr, \infty)$

Assume in addition $\mathrm{Vol}(M_j^3) \le V_0$
and $\mathrm{diam}(M_j^3) \le D_0$



$\Rightarrow$ a subsequence M_j^3 converges GH and $\mathcal{F}$ to M_∞ where M_∞ is a cantably H^3 rectifiable space.

(Note this works for any n, not just n=3).

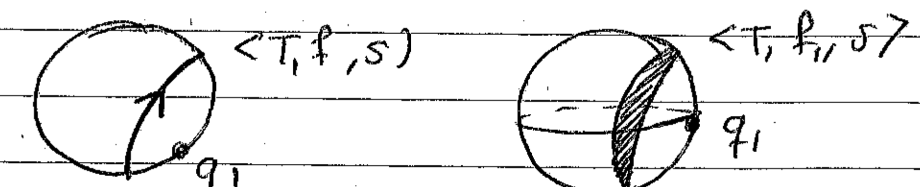


take infimum so going thru the fancy circle gives a closer guy.

Ambrosio - Kirchheim Slicing Theorem

Let $\langle T, f, s\rangle = (\partial T) \llcorner f^{-1}(s,\infty) - \partial(T \llcorner f^{-1}(s,\infty))$ (f Lipschitz)

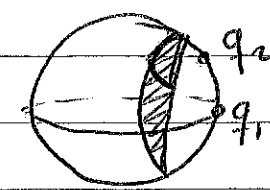
$T = [\![B_p(r)]\!]$ $f_1(x) = \rho_{q_1}(x)$



$$\int_{s \in \mathbb{R}} \mathbb{M}(\langle T, f, s\rangle)\, ds \le \mathrm{Lip}(f)\, \mathbb{M}(T)$$

Iterate

$< T, f_1, f_2, S_1, S_2 > = << T, f_1, S_1 >, f_2, S_2 >$



$\int \mathbb{M}(< T, f_1, f_2, s_1, s_2 >)\, ds_1 ds_2 \le \mathbb{M}(T)$

if $Lip(f_i) \le 1$

Sliced Filling Volume (Portegies-Sormani)

$SF(p, r, q_1, q_2) = \int_{t_1, t_2 \in \mathbb{R}} Fillvol(\quad) dt_1 dt_2$

$(\partial < [[B_p(r)]], \rho_{q_1}, \rho_{q_2}, t_1, t_2 >)$

$\le \int_{t_1, t_2 \in \mathbb{R}} \mathbb{M}([[B_p(r)]], \rho_{q_1}, \rho_{q_2}, t_1, t_2) dt_1 dt_2$

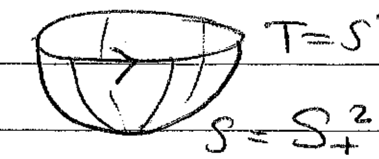
$\le Lip(\rho_{q_1}) Lip(\rho_{q_2}\, \mathbb{M}([[B_q(r)]]))$

THM $SF(p, r, q_1, q_2)$

continuous wrt $\mathcal{F}$ convergence

Gromov Filling Volume

$Fill\,vol(\partial T) = \inf \{ \underset{\text{weighted volume}}{\mathbb{M}(S)} : \partial S = \partial T \}$



$S \sim$ Wegner

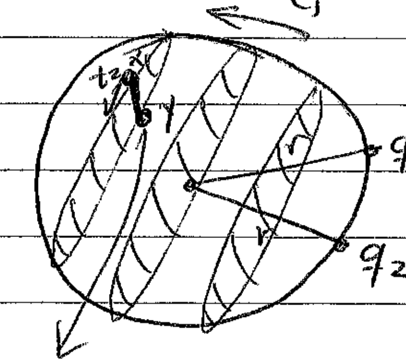
$M_j \xrightarrow{\mathcal{F}} M_\infty$

$Fillvol(\partial M_j) \to Fillvol(\partial M_\infty)$

THM $SF(p, r, q_1, q_2) \ge C(2\beta)^{m-1} r^m$

Why is this true?

this result is the one that prevents contractions.



slice by q_1 first then by q_2

$\partial < [[B_p(r)]], \rho_{q_1}, \rho_{q_2}, t_1, t_2 >$

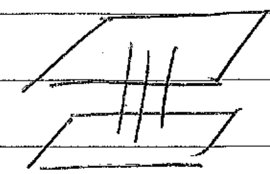
boundary is a pair of pts

$Fillvol([[x]] - [[y]]) \ge d(x, y) \ge Cr$

by tetrahedral property

only constants come from integral.

If you don't have the tetrahedral property



in a torus :

each slice is a circle

$\partial(\bigcirc) = 0$

Yi Wang - Finite total Q-curvature on a locally conformally flat manifold

joint work with Zhiqin Lu.

In conformal geometry, there is a notion of Q-curvature (Branson 1980's)

In dim 4

just a scalar that is paired w/ a conformal covariant operator

$$\rightarrow Q_g := \frac{1}{12}\left\{-\underbrace{\Delta R}_{\text{leading terms}} + \frac{1}{4}R^2 - 3|E|^2\right\}$$

E - traceless part of Ric

In other dimensions it has other expression

Paneitz op

$$D_g := \Delta^2 + \delta\left(\frac{2}{3}R_g - 2Ric\right)d$$

δ - divergence

d - differential

(lost her)