Open Questions

- Uniqueness of tangent cone at ∞. (should be there if Q-curvature is int or Q^+ is int)
- Geodesic distance estimate for $n \neq 2$
- Volume growth of geodesic balls.

Lookup (Gauss map $M \longrightarrow S^n$)

Raquel Perales Aguilar - Convergence of Manifolds and Metric Spaces with boundary

I. Gromov-Hausdorff convergence

DEFINITION (X_1, d_1), (X_2, d_2) compact metric spaces

check $d_{GH}((X_1, d_1), (X_2, d_2)) = \inf_Z \{ d_H^Z(\varnothing(X_1), \varnothing(X_2)) \}$

inf is over where $\varnothing_i : X_i \to Z$

where $d_H^Z(A, B) = \inf_\epsilon \{ \epsilon \mid T_\epsilon A \supset B, T_\epsilon B \supset A \}$

THEOREM Gromov (compactness theorem)

Let $\mathcal{X}$ denote the class of compact metric spaces s.t.

1) $\exists\, D > 0$ s.t. $\operatorname{diam}(X) \leq D \quad \forall X \in \mathcal{X}$

2) $\exists\, N(\;, \mathcal{X}) : (0, \infty) \longrightarrow \mathbb{N}$

~~where~~ $\forall X \in \mathcal{X} \;\exists\; \{ B(x_i, r) \}_{i=1}^{N(r)}$ cover of X for all $X \in \mathcal{X}$

then each sequence of $\mathcal{X}$ has a GH convergent subsequence

THEOREM (for manifolds w/ no boundary)

(M_j, g_j) of Riemann manifold

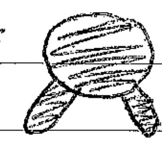
$\operatorname{Ric}(M_j) \geq K \quad \operatorname{Vol}(M_j) \leq V \quad \operatorname{diam}(M_j) \leq D$

then $\exists$ a subsequence of $\{M_j\}$ that GH-converges.

Key: Come up with N in order to apply the compactness theorem. We will use the Bishop-Gromov volume comparison.

Example: (with boundary)

$\mathbb{E}^2$ (Euclidean space

$M_1 =$  $M_2 =$  ....

$Ric(M_j) = 0 \quad Vol(M_j) \le V \quad diam(M_j) \le D$

but this sequence does not converge

If we have boundary we need to add more conditions.

Kodani: $|sec_*|, |sec_\partial| < K$

$0 < \mathrm{II} < \lambda \quad vol \le v \quad diam \le D$

$\Rightarrow$ Lipschitz precompactness

which then implies GH convergence

Anderson – etc.

Bounds the $|Ric|$, $|Ric_\partial|$ and several injectivity radius (ex volume, diam) and obtains $C^{1,\alpha}$ convergence

Wong – $Ric \ge r, \ diam \le D, \ \lambda^- \le \mathrm{II} \le \lambda^+$

GH precompactness theorem

Knox: $|sec|, |sec_\partial|, \ 0 \le \frac{1}{H_0} < H < H_0$

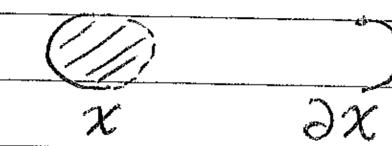
$vol(\partial M) > V$

and obtaines weak $L^{1,p}$ convergence

I.I Now our results.

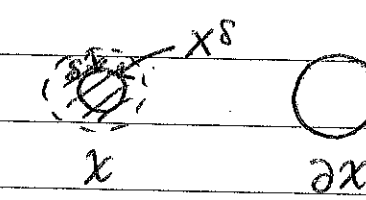
DEFINITION (X,d) is a metric space

$\partial X = \bar{X} \setminus X$

Ex: 

in euclidean space.

$X^\delta = \{ x \in X \mid d(x, \partial X) > \delta \}$

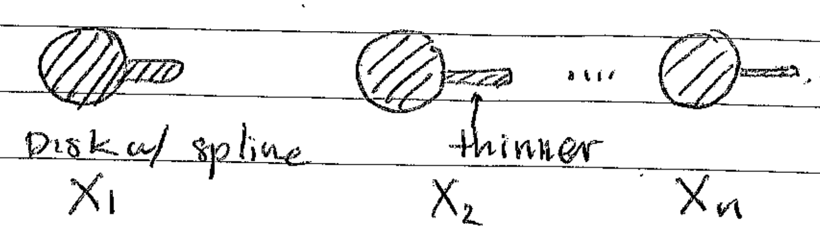
Ex: 

What we do is the following:

$X_1, \ X_2, \ X_3, \ \ldots$

assign $X_1^\delta, X_2^\delta, X_3^\delta, \ldots$

another sequence by fixing δ

Example 1: 

for large δ

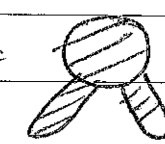
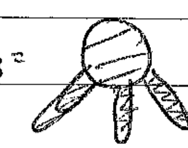
X_1^δ X_2^δ X_n^δ

Now we can study the convergence of X_n^δ
We don't want to impose any smooth conditions on boundary

THEOREM 1 (Perales Aguilar, Sormani)
Let $\mathcal{M}(n, \delta, D, \Theta, V)$ be the class of Riemannian manifolds with boundary such that

and $Ric \geq 0$
$Vol \leq V$

metric
$diam(M^\delta, d_{M^\delta}) \leq D \quad \forall M \in \mathcal{M}$
$\exists\, p \in M$ s.t. $vol(B(p,\delta)) \geq \Theta \delta^n \quad \forall M \in \mathcal{M}$.
$Vol(M) \leq V \quad Ric(M) \geq 0$
then for each sequence $\{M_j\}$ there $\exists$ subsequence $\{M_{j_k}^\delta\}$ that GH converges.

EX 2: $M_1 =$  $M_2 =$  $M_3 =$ 

For this sequence, $\forall \delta$ $\{M_j\}^\delta$ converges.
But $\{M_j\}$ does not converge.

Problematic sequences are those where $\delta \to 0$.
the closest to the boundary

(Perales Aguilar)
THEOREM 2: Under the hypothesis of thm 1
Suppose that δ_i is a sequence $\to 0$ and is decreasing. Let $D_i > 0$ s.t. $diam(M_j^{\delta_i}, d_{M_j^{\delta_i}}) \leq D_i$ and $\{(\partial M_j, d_j)\}$ GH converges.
$\Rightarrow \{M_j\}$ subconverges in GH sense
$\hookrightarrow$ ($\exists$ a subsequence that convergence)

Key: Come up with $N(, \{M_j\})$. to do that we use $N(, \{M_j^{\delta_i}\})$ and $N(\cdot, \{\partial M\})$

Intrinsic Flat Convergence

DEFINITION let (X, d, T) n-integral currents
(X,d) metric space
T n-integral current structure defined on X.
If (M,g) is compact and oriented, then $X = M$ and $d = d_g$ and $T =$ integration of top forms over M.
$T \to$ gives us a measure $\|T\|$

For (M,g) $\|T\|(A) = Vol(A)$. for $A \subset M$.
mass $:= \mathbb{M}(T) = \|T\|(M) = Vol(M)$.
Let (X_i, d_i, T_i) n-integral current spaces
Then $d_F(X_1, X_2) = \inf_Z \{ \mathbb{M}(U) + \mathbb{M}(V) \mid \phi_{1\#}(T_1) - \phi_{2\#}(T_2) = U + \partial V \}$
$\phi_i : X_i \to Z$ isometric embedding
U^n, V^{n+1}

Wenger: If (X_j, d_j, T_j) of n-integral current spaces such that
$$\mathbf{M}(T_j),\ \mathbf{M}(\partial T_j) \leq V \text{ and } \operatorname{diam}(X_j) \leq D$$
Then $\exists$ a sequence that converges in IF sense

So, to get IF convergence in theorem 2 we only need to add that
$$\operatorname{Vol}(\partial M_j) \leq A$$
But these two limits might not agree
(ie GH limit vs IF limit)

GH = IF for manifolds and metric spaces with no boundary.

Sormani – Portegies: Using tetrahedral property
Sormani – Wenger: this result is for manifolds that satisfy: $Ric \geq 0$, $Vol \leq V$, $Vol \geq \nu > 0$
Then GH = IF
Matveev – Portegies: $Ric \geq K$, $vol \leq V$, $vol \geq \nu > 0$

Li - Perales Aguilar: Alexandrov spaces

THEOREM 3: Under the conditions of THM 2

- If $\operatorname{Vol}(\partial M_j) \leq A \Rightarrow X \setminus X_\partial \subset Y$
  where X is the GH limit of $\{M_{j,k}\}$
  X_∂ is the GH limit of $\{\partial M_{j,k}\}$
  and Y is the IF limit of $\{M_j\}$
- If $X_\partial \subset Y$ then $X = Y$

Idea of proof:

- Sormani-Wenger If GH and IF limit exists $\Rightarrow Y \subset X$.
- To prove $X \subset Y$. Recall that
$$Y = \{x \in \bar{Y} : \liminf_{r} \frac{\|T\|(B(x,r))}{\omega_n r^n} > 0\}$$

For that we Fillvol b/c its continuous wrt IF distance and because $\|T\|(B(x,r)) \geq \operatorname{Fillvol}(\partial B(x,r))$.

Result on Alexandrov spaces:
(X_j, d_j, T_j) n-int current spaces
$(X_j, d_j) \in \operatorname{Alex}^n(O)$ s.t that T has weight $= 1$
curvature bounded above by O

then we have unif on diam and in the mass then either IF = GH limit or IF = zero integral current.