YANG-MILLS SINGULARITIES (joint w/ J. Streets)

BACKGROUND.

NOTATION

D_∇ : skew symmetrization
D^*_∇ : L^2 adjoint of ∇.
F_∇ : curvature tensor

(E,h) V.B
$\downarrow$
(M,g) Smth mfld, $\partial M = \phi$.

∇: h-compat connection.

YANG-MILLS ENERGY $\mathcal{YM}(\nabla) := \int_M |F_\nabla|^2 dV_g$

NEG GRAD FLOW $\frac{\partial \nabla_t}{\partial t} = -D^*_{\nabla_t} F_{\nabla_t}$

Fact: If YMF stops at $T \in \mathbb{R}$ $\exists X \in M$ s.t. $\lim_{t \nearrow T} |F_{\nabla_t}(x)| = \infty$.
Def$\underline{n}$: Let (X,T) be singular point of flow.

$$|F_{\nabla_t}(X)| < \frac{c}{(T-t)}$$

dim M	Result
2, 3	LTE & Convergence (Råde)
4	If $\exists (X,T)$ sing, then "bubbles" (Strüwe)
	$T<\infty \Rightarrow$ Sing. is type II (K–, Streets)
5+	Explicit type I sing. examples (Naito, Grotowski)

"The analysis of Harmonic maps & their heat flows. Lin, Wang

GOAL: Study structure of singular set for $n \geq 4$ via HMHF techniques.

Def$\underline{n}$ For $0 \leq \ell \leq 2+\dim M$ and $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$, the ℓ-dim'l Parabolic Hausdorff measure of Ω is

$$\mathcal{P}^\ell(\Omega) = \lim_{\delta \searrow 0} \mathcal{P}^\ell_\delta(\Omega) = \lim_{\delta \searrow 0} \inf \left\{ \sum_i r_i^\ell \;\middle|\; \Omega \subseteq \bigcup_i P_{r_i}(z_i),\ z_i \in \Omega,\ r_i \leq \delta \right\}$$

roughly speaking cover in small balls but compute ℓ-dim area

Generalization of euclidean measure
differentiates sets on higher lvl

where $P_r(z_0) := \{ z = (x,t) \in \mathbb{R}^n \times \mathbb{R} \mid |x - x_0| < r,\ |t - t_0| < r^2 \}$.

MAIN RESULT I

Weak Compactness Thm (rough statement)

Wk H_1^2 limits of smth solns to YMF are wk solns to YMF which are smth outside a closed set Σ with $\mathcal{P}^{n-2}(\Sigma) < \infty$.

parabolic Hausdorff measure

Then, clarify the obstruction to strong convergence ("bubbing off")

MAIN RESULT II

Weak to Strong Compactness. A sequence of YMF solns converging wkly in H_1^2 either

① converges strongly in H_1^2, and $\mathcal{P}^{(n-2)}(\Sigma) \equiv 0$.
② ∇_t admits a blowup limit which is YM on S^4

COROLLARY I

For $E \to M^n$, $n \geq 4$, ∇_t smth YMF soln on $[0,T)$ s.t. $\lim_{t \nearrow T} |F_{\nabla_t}|_{C^0} = \infty$. Then $\exists$ sequences $\{(x_i, t_i)\} \subseteq M \times [0,T)$, $\{\lambda_i\} \subseteq \mathbb{R}$ s.t. the corresponding BU seq converges (modulo gauge tfm) to either a

① YM conn. on S^4 ← YM in $\mathbb{R}^4$ & trivial in $\mathbb{R}^{n-4}$ ← YM in other dim
② YM soliton

OUTLINE

I Entropy def's & monotonicity

II Weak compactness

III Structure of limit measures

IV Stratification and blowup

I Entropy Def$^{\underline{n}}$'s & Monotonicity

For $z_0 := (x_0, t_0) \in \mathbb{R}^n \times [0,\infty)$ $\quad G_{z_0}(x,t) = \dfrac{e^{-\frac{|x-x_0|^2}{|t-t_0|}}}{(4\pi|t-t_0|)^{n/2}}$ (Euclidean Backwards Heat Kernel)

ϕ cutoff : $\phi \in [0,1]$, $\phi = 1$ on $B_{i_M/2}(x_0)$, $\operatorname{supp}\phi \subseteq B_{i_M}(x_0)$.

Set

$$\Phi_{z_0}(R;\nabla_t) := R^4/2 \int_{M \times \{t_0 - R^2\}} |F_{\nabla_t}|^2 \phi^2 G_{z_0}\, dV$$

$$\Psi_{z_0}(R;\nabla_t) := R^2/2 \int_{\underline{M \times ([t_0-4R^2, t_0-R^2] \cap (0,\infty))}} |F_{\nabla_t}|^2 \phi^2 G_{z_0}\, dV\, dt$$

$\rightarrow T_R(t_0) \leftarrow$ we'll use this later.

Motivations to use
- G_{z_0} focuses about z_0 ("localizes")
- Both exhibit monotonicity WRT R, t
- critical pts related to self-shrinker

Entropy Monotonicity

Thm of Hong & Tian

[∇_t is YMF soln. For $z_0 \in M \times [0,T]$ and R_1, R_2 s.t. $0 < R_1 \leq R_2 \leq \min\{i_M, \sqrt{t_0}/2\}$,

$$\underset{\downarrow\,\Phi}{\Psi_{z_0}}(R_1;\nabla_t) + \int_{R_1}^{R_2} r \int_{\underset{\downarrow\, M\times\{t_0-R^2\}}{M \times ([t_0-4R^2, t_0-R^2] \cap (0,\infty))}} |t-t_0| \left| \frac{x-x_0}{2|t-t_0|} \lrcorner F_{\nabla_t} - D^*_{\nabla_t} F_{\nabla_t} \right|^2 \phi^2 G_{z_0}\, dV\, \underset{\downarrow\,\times}{dt}\, dr$$

$$\leq e^{C(R_2-R_1)} \underset{\downarrow\,\Phi}{\Psi_{z_0}}(R_2;\nabla_t) + C(R_2 - R_1)\, YM(\nabla_0)$$

in Euclidean case $\overset{\text{with } \phi \equiv 1}{\text{strict}}$ R-monotonicity

Lem (KS) "Φ sandwich"

[∇_t YMF soln on $[0,T)$] $\exists$ uniform $C > 0$ s.t. for $z_0 \in M \times [0,T)$ & R s.t. $R \in (0, \min\{i_M, \sqrt{t_0}/2\}]$,

$$C^{-1}\Psi_{z_0}(R;\nabla_t) \leq \Phi_{z_0}(2R;\nabla_t) \leq C\,\Psi_{z_0}(2R;\nabla_t).$$

ε - REGULARITY

Thm (Hong, Tian)

Idea of thm: control over $\Psi \Rightarrow$ ptwise control of curv on Parabolic balls

[∇_t YMF soln on $[0,T)$]

$\exists C, \delta, \varepsilon_0 > 0$ depending on (M,g) & $YM(\nabla_0)$ s.t. for $z_0 \in M \times [0,T)$ and $0 < R < \min\{2_M, \sqrt{t_0}/2\}$ satisfying $\Psi_{z_0}(R; \nabla_t) < \varepsilon_0$, one has

$$\sup_{P_{\delta R}(z_0)} |F_{\nabla_t}|^2 \leq \frac{C}{\delta R^4}$$

II Weak compactness

[Precise statement]

Thm (KS)

$$\left[\begin{array}{l} \{\nabla_t^i\} \text{ seq of YMF smth solns on } [-1,0] \\ YM(\nabla_t^i) \leq YM(\nabla_{-1}^i) < C \\ \{\nabla_{-1}^i\} \xrightarrow{\text{wkly}} H^{1,2}_{loc}(\mathcal{A}_E(M)) \end{array}\right. \qquad \left\{\begin{array}{l} \bullet\ \nabla_t^i \to \nabla_t \ (L^2_{loc}(M \times [-1,0])) \\ \bullet\ \frac{\partial \nabla_t^i}{\partial t} \xrightarrow{\text{wkly}} \frac{\partial \nabla_t}{\partial t} \ (L^2_{loc}(M \times [-1,0])) \\ \bullet\ F_{\nabla_t^i} \xrightarrow{\text{wkly}} F_{\nabla_t} \ (L^2_{loc}(M \times [-1,0])). \end{array}\right.$$

∇_t is gauge equiv to YMF wk soln, & $\exists\, \Sigma$ (closed) w/ locally finite $(n-2)$ dimensional (★) parabolic Hausdorff measure s.t. ∇_t is a smth YMF soln on $(M \times (-1,0)) \setminus \Sigma$.

sketch of pf.

$$\text{set} \quad \Phi^i_{z_0}(r) := \begin{cases} \Phi_{z_0}(r; \nabla_t^i), & r \in (0, \sqrt{1+t_0}) \\ \Phi_{z_0}(\sqrt{1+t_0}\,; \nabla_t^i) & \text{o.w.} \end{cases}$$

$$\Sigma := \bigcap_{r>0} \{ z \in M \times [-1,0] \mid \lim_{k\to\infty} \Phi^k_z(r) \geq \varepsilon_0 \}$$

capturing pts with a quantum of energy

from ε-regularity thm

Sketch of (★)

We take K cmpt, apply Vitali's covering lemma so $\{P_{r_k}(z_k)\}_{k=1}^{\ell}$ disjoint and $K \cap \Sigma$ covered by $\{P_{5r_k}(z_k)\}_{k=1}^{\ell}$

apply approximations via ε-regularity, "sandwich thm, etc. so

$$\varepsilon_0 \leq \Phi^k_{z_k}(\delta r_k) \leq \cdots \leq \underbrace{\frac{e^{-1/(4\delta)^2}}{4\delta^n} C\, YM(\nabla_{-1})}_{\text{Pick } \delta>0 \text{ so this } \leq \frac{\varepsilon_0}{2}} + \underbrace{C_\delta r_k^{2-n} \int_{P_{r_k}(z_k)} |F^k|^2 dV dt}_{\leq \frac{\varepsilon_0}{2}}$$

$\Rightarrow$

So $r_k^{n-2} \leq \frac{C}{\varepsilon_0} \int |F_t^k|^2 dV\, dt$.

Then compute.

$$\mathcal{P}^{n-2}_{5\delta}(P_R \cap \Sigma) \leq \sum_{k=1}^{\ell} (5r^k)^{n-2} \leq \left(r_k^2 \int_{\bigcup_{k=1}^{\ell} P_{r_k}(z_k)} |F_t^k|^2 dV dt\right)$$
$$\leq C\, YM(\nabla_{-1}).$$

Send $\delta \to 0$ Result follows! □

Next we need to refine the analysis to find out what inhibits strong convergence.

To do so we analyze blowups & in particular tangent measures to analyze Σ.

III. Structure of limit measures

Idea: instead of blowup of seq in stronger convergence cases use measures to "see" more.

The measures

$$\left\{ |F_{\nabla_{t}^{i}}|^2 \, dV \, dt \right\} \text{ and } \left\{ \left|\frac{\partial \nabla_t^i}{\partial t}\right| dV \, dt \right\}$$ admit subseq which converge in Radon measure to limit.

$$|F_{\nabla_t^i}|^2 dx\, dt \to |F_{\nabla_t^\infty}|^2 dx\, dt + \nu \equiv \mu$$

defect measure; tells you how far from good convergence you have.

Lemma (KS)
[$z = (x,t) \in M \times [0,\infty)$

$$\Theta(\mu; z) := \lim_{R \searrow 0} R^2 \int_{T_R(z)} \phi^2(x)\, G_z(x,t)\, d\mu(x,t)$$

μ has a density Θ

exists & is upper semicontinuous $\forall z$. Moreover
$\Sigma = \{ z : \varepsilon_0 \leq \Theta(\mu, z) < \infty \}$.

alternate interpretation of Σ! Key to slicing apart

Def$^{\underline{n}}$
Parabolic Dilation: $P_{z_0,\lambda}(x,t) := \left(\frac{x - x_0}{\lambda}, \frac{t - t_0}{\lambda^2} \right)$

Apply to a measure via $P_{z_0,\lambda}(\mu) := \lambda^{2-n} \mu(P_{z_0,\lambda}(A))$
$A \subseteq \mathbb{R}^{n+1}$

WARNING: This law reflects scaling props of YMF densities, not in Euc. measure

Def$^{\underline{n}}$
construct tangent measure: take $\lambda_i \searrow 0$ & set

$$\mu^* = \lim_{i \to \infty} P_{z_0,\lambda_i}(\mu).$$

Lemma (KS)
[$z_0 \in \Sigma$, $\mu^* \in T_{z_0}(\mu)$
$\mu^* \llcorner \mathbb{R}^n \times (-\infty, 0)$ is invariant under parabolic dialation:

Weird notation "$\llcorner$" => "restriction".

$$P_\lambda(\mu^* \llcorner \mathbb{R}^n \times (-\infty,0)) = \mu^* \llcorner \mathbb{R}^n \times (-\infty,0).$$

Tangent Measures: Application

For any $z_0 \in M$ & μ^*, a tangent measure, we set

$$M(\Theta(\mu^*)) := \{ z \in \mathbb{R}^{n+1} : \Theta(\mu^*, z) = \Theta(\mu^*, 0) \}$$ ← density is same as origin

$$V(\Theta(\mu^*)) := M(\Theta(\mu^*)) \cap \mathbb{R}^n \times \{0\}$$ ← time zero slice.

Rmk
Parabolic scaling invariance => M, V linear subspaces, & $M = V$ or $M = V \times (-\infty, 0]$.

intuitively soliton. YM conn using wk blowups so cant say for sure

Def$^{\underline{n}}$

for $z_0 \in \Sigma$ & μ^* a tangent measure formed at z_0, set

$$\dim(\Theta(\mu^*)) := \begin{cases} \dim(V(\Theta(\mu^*))+2 & \text{if } M(\Theta(\mu^*)) = V(\Theta(\mu^*)) \times \mathbb{R}_{<0} \\ \dim(V(\Theta(\mu^*)) & \text{otherwise} \end{cases}$$

i.e. tangent measure

This means we have existence of parabolically scaling invariant tangent cones

IV STRATIFICATION AND BLOWUP

We STRATIFY SINGULAR SET VIA B.WHITE METHODS

Thm for $k \in [0,n] \cap \mathbb{N}$ set

$$\Sigma_k := \{ z_0 \in \Sigma : \dim(\Theta^\circ(\mu^*, \cdot)) \leq k, \ \forall \mu^* \in T_{z_0}(\mu) \}.$$

Then $\dim_P(\Sigma_k) \leq k$ and Σ_0 is countable

in simplest setting Σ_k are mflds of these dimensions

We can refine the tangent measure Blowup to get characterization of strong H^1_2 obstruction.

Thm (K-S)

[$\{\nabla^i_t\}$ seq. of smth YMF solns on $[-1,0]$ converging weakly ← write weak conv.

Then either

(A) $\exists$ blowup sequence converging to YM conn on S^4

(B) OR

$$|F_{\nabla^i_t}|^2 dV\, dt \longrightarrow |F_{\nabla^\infty_t}|^2 dV\, dt \text{ converging as Radon measures.}$$

So $\{\nabla^i_t\} \to \nabla^\infty_t$ strongly in H^2_1

So ∇^∞_t is weak soln to YMF, $\mathcal{P}^{n-2}(\Sigma) = 0$.

Sketch of
Proof

Idea: failure of strong H^2_1 convergence $\Rightarrow$ defect measure ν nontrivial

see by showing tan measures have full dimensionality

$\exists$ linear subspace P (n-4 dim) s.t. $\forall t > 0$, $\operatorname{supp} \mu^*_t = P$.

then we split into P & $P^\perp$ and show solution is "gaining invariance in the P and time directions