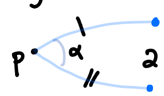
Riemannian Manifolds with Lower Curvature Bound

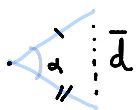
M : complete Riemannian mfld
$\sec M$: sectional curvature, assume $\sec M \geq k$, $k \in \mathbb{R}$
Let's consider various geometric ways to interpret this condition (comparison thm)
S_k^2 : Constant curvature k-surface, simply connected — $k=0$ Euclidean, $k=1$ Sphere, $k=-1$ hyperbolic plane

(V) "Hinge" version (angle + geodesic)



2 geodesics — consider distance d between endpts

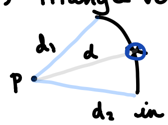
triangles are "fatter" in this space



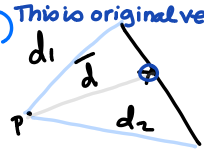
$d \leq \bar{d}$ in S_k^2

(T) Triangle version (geodesic + distance) — This is original version

"O" marks corresponding pts.



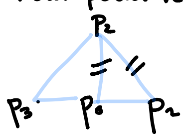
geodesic triangle in M



$d \geq \bar{d}$ in S_k^2

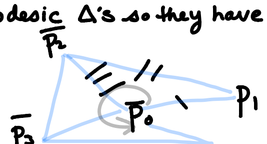
expressed <u>without</u> angles but with distances

(Q) Four point version



$\longrightarrow$ draw corresponding geodesic Δ's so they have adjacent sides

constant curvature triangle

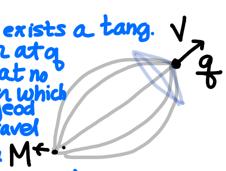


condition : sum of angles is $\leq 2\pi$ (these are "comparison angles").
$\Sigma \leq 2\pi$ in S_2^k

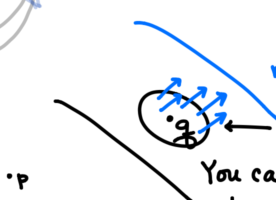
No curves or angles necessary, only need metric in this case!
curv $\geq k$!
we reserve "sec" for Riemannian manifolds

Critical Pt Theory for dist functions.

there exists a tang. vector at q so that no matter which min. geod you travel on you make angle $\geq 2\pi$.



a pt. q is regular for dist_p.
all minimal directions from q to p are in some cone
q is critical if not regular



take small nbhd about q, locally pts are regular — extend v.f. in nbhd
You can construct a local vector field (like Morse theory)
<u>Key point</u> : you have an isotopy.

Applications (3 to consider)

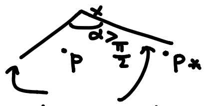
Thm (Diam Sphere Thm) (G & X) $\sec M \geq 1$, $\text{diam}\, M > D_2$ ← diam of 2-unit sphere (diam M: maximal dist between any 2 pts)
$\Downarrow$
$M \approx S^n$ ($\approx$: homeomorphic)

Thm (Betti Number Thm) (Gromov) $\sec M^n \geq k$, $\text{diam}\, M \leq D$ — NOTE (D, k, n) all real numbers
$\Downarrow$

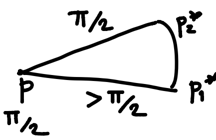
$\dim H_*(M;F) \le C(n,k,D)$
Bound on how complicated Homology of mfld is.

Thm (Homotopy Finiteness) (G & P) Given n, k, D and v, there are at most finitely many homotopy types of M^n with $\sec M \ge k$, $\operatorname{diam} M \le D$, $\operatorname{vol} \ge v$.

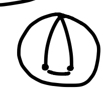
Consider $\sec M \geq 1$, $\operatorname{diam} M > \pi/2$.



take any min geod, α is always $> \pi/2$



claim p^* is unique



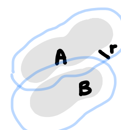
Suppose not

$\operatorname{dist}(p, p^*) = \operatorname{diam} > \pi/2$ only on p^*

For Homotopy Finiteness Thm proof us Gromov Hausdorff distance for closeness

Gromov Hausdorff distance d_{GH}.

given Z: cpt metric space. A, B closed $\subseteq Z$.



r-neighborhoods

The HAUSDORFF METRIC

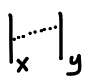
$$d_H^Z(A,B) = \inf\{ r \mid D(A,r) \supseteq B,\ D(B,r) \supseteq A \}$$

$D(\cdot, r)$ is the "r-neighborhood".

$$d_{GH}(X,Y) := \inf_{Z:\, X, Y \subseteq Z} d_H^Z(X,Y)$$

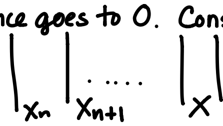
If 2 spaces have distance 0, they are isometric

Note: it suffices to choose $Z = X \amalg Y$ (the disjoint union).



Q: what does $X_n \xrightarrow{GH} X$ mean?

GH distance goes to 0. Construct metric on disjoint union and look at convergence wrt Hausdorff distance.



Example $X = pt$, $Y = \vdots\,\}\,1$ ← space of 2 pts dist 1 apart $\quad d_{GH}(X,Y) = 1/2$

take disjoint union as before & define dist.

$Y = \overset{1}{\cdot}\,\overset{\cdot}{_1}\,\overset{1}{\cdot}$ ← space of 3 $\quad d_{GH}(X,Y) = 1/2$

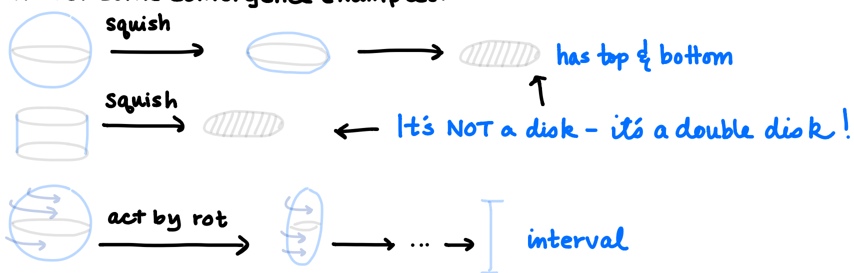
Any compact metric space is as close as you want to a finite metric space
this is a very coarse topology.

Theorem of Gromov → relatively compact metric spaces

$\mathcal{X}$ class of compact metric spaces is GH precompact if $\exists C(\varepsilon)$: Any $X \in \mathcal{X}$ can be covered by $\le C(\varepsilon)$ - ε-balls. (The number is uniform!)

Let's consider some convergence examples.

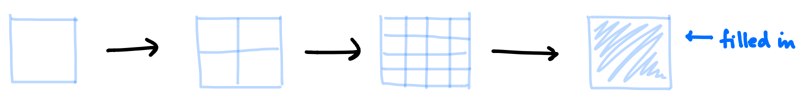
Ex



Now look at not-so-nice examples

attach more and more handles which get smaller each iteration

1 dim



topology changes drastically, curvature $\searrow -\infty$

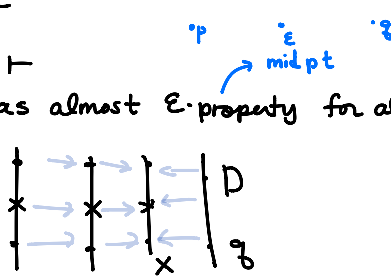
Preserved properties

X_n length space

$\Downarrow$

$\lim X_n = X \quad \dashv \vdash$

length space $\Leftrightarrow$ has almost ε-property for all pts



Def$^{\underline{n}}$ Alexandrov Space X

(1) X length space
(2) $\exists k$: curv $X \geq k$
(3) $\dim_H X < \infty$
  ↳ Hausdorff dim

Consequently if M_n has curv $M_n \geq k$ then $M_n \xrightarrow{GH}$ (Alexandrov space).

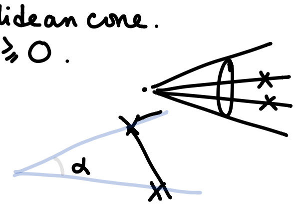
Ex $M = X$, sec $M \geq k$, $X = \lim_{GH} M^n$, sec $M \geq k$ ← all length spaces

$\Omega \subset_{\text{convex}} \mathbb{R}^n$, Ω & $\partial\Omega$ Alexandrov, curv ≥ 0.

Ex E : Alexandrov w/ curv ≥ 1.

$X = C_0 E$ Euclidean cone.
Alex. curv $X \geq 0$.

You can do hyperbolic cone too!



Say X is Alexandrov, $\text{curv}\, X \geqslant 0$.
Consider a submetry (generalization of submersion)
R-balls map ONTO R-balls.

$$X \xrightarrow{\pi} Y$$

$\text{curv}\, Y \geqslant k$, Y is also an Alexandrov space.

Structure of Alexandrov spaces
Surprisingly not so bad! Infitesmal $p \in X$, T_pX tangent cone

$\frac{1}{\varepsilon} B(p,\varepsilon)$ $\quad \lim_{GH\ \lambda \to \infty} \lambda(X,p)$.

You blow up about p, curvature $\to 0$, becomes scale invariant

$T_p X = C_o S_p X$ $\qquad \text{Curv}\, S_p X \geqslant 1$
space of directions at p

What about local structure

Thm (Perelman) $\forall p \in X$, $\exists\, \varepsilon > 0$ s.t. $\quad B(p,\varepsilon) \underset{\text{homeo}}{\sim} T_p M$.
proved by inverse induction on functions

local structure $\Rightarrow$ deep understanding of topology

All these Alex. space preserving transformations have many applications

Thm (Stability Thm) Perelman
Given X, $\text{curv}\, X \geqslant k$, $\exists\, \varepsilon = \varepsilon(X)$ s.t.
$d_{GH}(Y,X) < \varepsilon$, Y^n curv $Y \geqslant k$
$\Downarrow$
$X \sim Y$
homeomorphic.

MSRI Talk I

AIM @ 50 min

1_1

Manifolds with lower sectional curvature bounds and Alexandrov Geometry

① $\sec M \geq k$

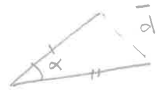
- (M, g) complete Riem. n-manifold.
- S^2_k simply connected, constant curvature k

Equivalent (Alexandrov Toponogov)

[$\measuredangle$ + geod.] (V)

in M

$d \leq \bar{d}$



in S^2_k

"Hinge" version

(classical $\triangle$ version) ⟵ SAY IN WORDS

[geod + dist] (T)

in M

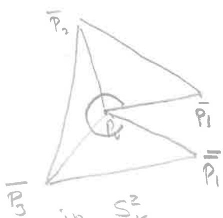
$D \geq \bar{D}$

in S^2_k

"Distance version"

[only dist] (Q)

$\operatorname{curv} X \geq k$



$\Sigma \tilde{\measuredangle} \leq 2\pi$

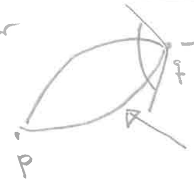
in S^2_k

((E) Embedding version (Buyalo?))

$\leq$ 6 minutes

(2) Critical Points for dist_p

Regular

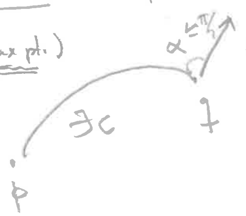


"regular" direction

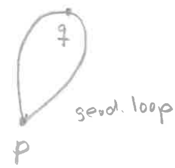
all min geod. from q to p make angle to $v > \pi/2$.

q critical if not regular :

(e.g. a max pt.)



example



Note q regular $\Rightarrow$ all $q' \in B(q, \varepsilon)$ regular

pa

convex combination of reg. directions is regular

$\Downarrow$ partition of 1 yield "gradient like" vectorfield

only regular point between two levels

Isotopy :

KEY

Can use any A cpt in place of p

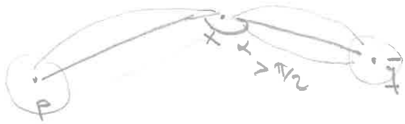
dist_A

≈ 6 minutes

13

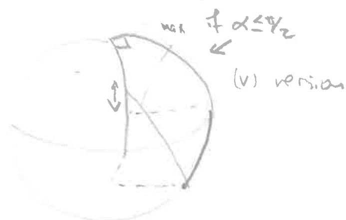
③ Applications (pre. Alex)

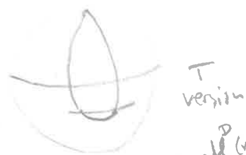
- Diam SphereThm (G- Shiohama)
  $\sec M \geq 1$, $\operatorname{diam} M > \pi/2$
  $\Downarrow$
  $M^n \sim S^n$ (topologically)



$|p\bar{q}| = \operatorname{diam} M$

$\bar{q}$ unique! by Toponogov





$M \in \mathcal{M}^D_k(n)$

- Betti # Thm (Gromov)
  $\exists\, C = C(n,k,D)$: $\sec M^n \geq k$, $\operatorname{diam} M \leq D$
  $\Downarrow$
  $H_*(M)$ gen. by $\leq C$ elements

Related Key observation : Weak Soul thm.

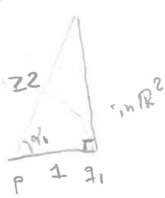
$\sec M \geq 0$, noncpt $\Rightarrow$ Finite top. type

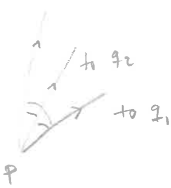
$M \simeq B(p,R) \subset M$

In fact $\exists R$: no crit pt. outside $B(p,R)$

or else $\exists\, q_1, q_2, \ldots$ $|p\, q_{i+1}| \geq 2|p\, q_i|$ all p-critical

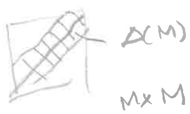
$\Downarrow$



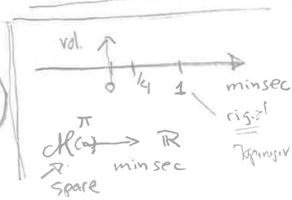


all angles at $p \geq$ some α_0

- Homotopy Finiteness $\exists\, C = C(n,k,D,v)$: $\mathcal{M}^D_{k,v}(n)$ $\operatorname{vol} M \geq v$ contains at most C homotopy types



≈ 7 minutes

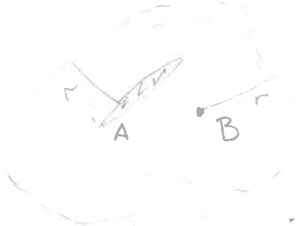


4 GH - metric

Z cpt metric

A, B ⊂ Z closed, $d_H^Z(A,B) = \inf\{r \mid B(A,r) \supset B, B(B,r) \supset A\}$

Hausdorff



X, Y cpt metric spaces : $d_{GH}(X,Y) = \inf_{X \subset Z \supset Y} d_H^Z(X,Y)$

suffices to take

$Z = X \amalg Y$!

$\boxed{\text{Metric on isom. classes}}$

Example $d_{GH}(pt, \because) = 1/2$

$X_n \xrightarrow[GH]{} X \iff \exists$ metric on $X \amalg_1^\infty X_n = Z$ s.t. $d_H^Z(X_n, X) \to 0$

Coarse : $\overline{\mathcal{F}}^{GH} = \mathcal{C}$ all cpt metric space

finite

$\forall \varepsilon \; \exists F$ ε-close.

THM Gromov $\mathcal{P}$ precompact $\Updownarrow$ $\exists C(\varepsilon)$ cover function, i.e. unif. # ε-balls needed on all element of $\mathcal{P}$

- Example $\mathcal{M}_K^D$ [$Ric \geq (n-1)K$ suffices.] Bishop Gromov rel. volume

- Examples

10 min

- Preserved properties

- X_i length. $\Longrightarrow$ $\lim X_i = X$ length

  p ε-midpt. q ✓

2 min

- $\operatorname{curv} X_i \geq k$ $\Longrightarrow$ $\operatorname{curv} X \geq k$, $X = \lim X_i$

  ∂-prop.

5 Alexandrov Spaces

(a) X length space

(b) $\operatorname{curv} X \geq k$

[(c) $\dim_H X < \infty$]

Examples & Constructions

- M Riem. man., $\sec M \geq k$
- $X = \lim_{GH} M_i^n$ $\quad \sec M_i^n \geq k$
- $\Omega \subset \mathbb{R}^n$ convex, $\partial\Omega$ $\operatorname{curv} \geq 0$

4 min

- $\operatorname{curv} E \geq 1 \Rightarrow C_0E$ has $\operatorname{curv} \geq 0$

  $C_0E = E \times [0,\infty) / E \times \{0\}$

$\Big(C_{-1}E \;,\; C_1E = S_1E \Big)$

[• $\operatorname{curv} E_i \geq 1 \Rightarrow \operatorname{curv} E_1 * E_2 \geq 1$: $C_0E_1 \times C_0E_2 = C_0(E_1 * E_2)$]

- $X \cup Y$ $\quad \partial X \equiv \partial Y$ ✓

- $X \xrightarrow{\pi} Y$ submetry $\quad \operatorname{curv} X \geq k \Downarrow \operatorname{curv} Y \geq k$.

Total 6 min page.

(6) Structure

- Infinitesimal $x \in X$ $\frac{1}{\varepsilon} B(x,\varepsilon)$ unit ball in scaled space.

  $T_x X = \lim_{\lambda\to\infty} (X, x)$

  $\text{curv}\, T_x X \geq 0$ scale invar.

  so $T_x X = C_0\, S_x X$ ← "unitsphere" space of dir

  $\text{curv}\, S_x X \geq 1$

Note   Any E with curv $E \geq 1$ (pos. curv.) $\exists$ space of dir.

Also $S_x X = \overline{G_x X}$ ← geodesic directions

- Local $\exists\, \varepsilon = \varepsilon(x)$: $B(x,\varepsilon) \simeq T_x X$ (P) homeo

  crit pt. theory!   Highly non-trivial. (start with n-fn. induction).

- Global structure : stratified into manifolds

  (Metric or top. ways -----)

- STABILITY THM (P) $\forall\, \varepsilon = \varepsilon(X)$, curv $X \geq k$, dim $X = n$

  $d_{GH}(X, Y) < \varepsilon$, curv $Y \geq k$, dim $Y = n$

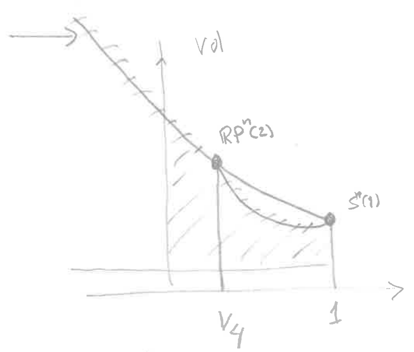
  $\Downarrow$ $Y \sim X$ homeo.!

Corollary $\mathcal{M}^D_{k,v}(n)$ contains at most finitely many homeo types

$\overline{\mathcal{M}^D_{k,v}(n)}$ cpt, $X \in \overline{\mathcal{M}}$ dim $= n$!   8 min

7 Additional Applications

Note $\operatorname{rad} X \le \operatorname{diam} X \le 2 \operatorname{rad} X$



say $\operatorname{Rad} M = \pi$

(3 min)

Vol Recognition

$\exists M$ $\operatorname{vol} M^n \sim \operatorname{vol} D_K^n(\pi)$, $\sec M \ge k$, $\operatorname{Rad} M = \pi$ $(k \le 1/4)$

such M are diffeo to S^n, $\mathbb{RP}^n$

& almost DO to ?

GP & PW
homeo / S \ diffeo
no sea equivalent

Diff. Stability Problem

1 $\sec M_i \ge k$ $M_i^n \to X^n$

$\Downarrow$ $M_i \sim M_j$ diffeo i, j large ?

(+) $\Rightarrow$ (a) Diffeo finiteness for $\mathcal{M}^D_{k,v}(n)$ also $n=4$

1 Pro-Wilhelm promising work

Σ

(b) $\sec M \ge 1$, $\operatorname{diam} M > \pi/2$ diffeo S^n ?

(note collapse D present $\operatorname{diam} \to \pi$)

3 also Wilking)

$\exists$ pair at dist $> \pi/2$ $\Rightarrow$ $(q+1)$ handle body ?

$\to$ $(n-2)$ such pair $\Rightarrow$ diffeo ?

9 minutes

8 COLLAPSE ?

18

Nothing Known ?! when X has singularities ?!

Problem $\quad M_i^n \xrightarrow[GH]{} X^\ell \quad \ell < n$

singular.

4 min

(1) Restrictions on X ?

(2) Almost Submetry ?, strata fibration ?

(3) $X = pt$ $\quad M$ almost non-curved
$\quad \cap$
$\quad \mathcal{P}_0$- $\quad$ Nilpotent

Extended Bott-conjecture

Any $M \in \mathcal{P}_0$- $\exists$ topologically elliptic

i.e. $\{\beta_k(\Omega M)\}$ grows at most polynomially

Orbitspaces, when $\boxed{\sec M \geq 0}$

$G \times M \to M$, $\quad M \to M/G$ very nice structure
(Alex + + + -)

$\sec M \geq 0$
$\Rightarrow M/G \geq 0$

6 min

(HK - GW) $\quad M^4$ 1-conn. cpt. , $\sec M \geq 0$

$\Downarrow \; S^1 \subset \mathrm{Iso}(M)$

$(M, S^1) \sim$ equiv diffeo to lin action on $S^4, \mathbb{C}P^2$

$\mathrm{Fix}\, S^1$ 0 or 2-dim. $\chi \leq 4$

or sub $S^1 \subset T^2$ where

Galaz-Garcia Kerin $\quad T^4 \curvearrowright S^3 \times S^3 \downarrow /T^2$

either $S^2 \times S^2$, $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$

$S^1 \subset T^2 = T^4/T^2 \curvearrowright M^4 \leftarrow$

Orlik C
Raymond

purely geometric

e.g. S^4 knotless

2-fold branched cover S^3, $|\pi_1(S^3_2)| \geq 3$ $\nexists$ knotted

$M^4/S^1 = S^3$ top. (Perelman) - or, Hamilton ?

Dinkelbach-Leeb