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Ricci Curvatı Comparison Geometry

Laplace Comparsion Proof of the

Comparison

of Laplaciar Comparison

Volume Compariso

Two Points Distance Second

Comparisor

Eigenvalue Comparison

Comparison Results for Ricci Curvature I

Guofang Wei

UCSB, Santa Barbara

Introductory Workshop: Modern Riemannian Geometry, MSRI, 01/18/16



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Ricci Tensor

Given

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a Riemannian manifold
$$(M'',g)$$
, the Ricci tensor
 $\operatorname{Ric}(X,Y) = \sum_{i=1}^{n} \langle R(X,e_i)e_i,Y \rangle$

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First Neuman Eigenvalue Comparison - the trace of curvature tensor. Symmetric 2-tensor.

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Ricci Tensor

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Introduction Ricci Curvature Comparison Geometry

Comparsior Proof of the Laplacian Comparison

of Laplacia Compariso Volume Comparison Two Point Distance Second Derivative

First Neumai Eigenvalue Given a Riemannian manifold (M^n, g) , the Ricci tensor $\operatorname{Ric}(X, Y) = \sum_{i=1}^n \langle R(X, e_i) e_i, Y \rangle$

- the trace of curvature tensor. Symmetric 2-tensor.

Italian mathematician: Gregorio Ricci-Curbastro (1853-1925)



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Ricci Curvature

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> First Neumani Eigenvalue Comparison

Ricci curvature for |X| = 1 is

$$\operatorname{Ric}(X,X) = \sum_{i=1}^{n} K(X,e_i)$$

- the "average" of sectional curvature

Ricci Curvature

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Ricci curvature for |X| = 1 is

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$$\mathsf{Ric}(X,X) = \sum_{i=1}^{n} K(X,e_i)$$

- the "average" of sectional curvature

For \mathbb{S}^n , $K \equiv 1$, Ric $\equiv n - 1$.

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Ricci Curvature

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Ricci curvature for |X| = 1 is

$$\operatorname{Ric}(X,X) = \sum_{i=1}^{n} K(X,e_i)$$

- the "average" of sectional curvature

For \mathbb{S}^n , $K \equiv 1$, Ric $\equiv n - 1$.

Ricci curvature is in between sectional curvature and scalar curvature, measures the deviation of the volume element from the Euclidean one.

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Vacuum Einstein equation

ntroduction **Ricci Curvature** Comparison Geometry

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First Neumar Eigenvalue Comparison $\operatorname{Ric} = \lambda g, \quad \lambda \text{ constant}$

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Vacuum Einstein equation

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First Neumar Eigenvalue $\operatorname{Ric} = \lambda g, \quad \lambda \text{ constant}$

Ricci flow

$$\frac{\partial g}{\partial t} = -2$$
Ric.

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Vacuum Einstein equation

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Ricci flow

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Related to Optimal Transportation

Model Spaces

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Comparison

First Neumani Eigenvalue Comparison M_{H}^{n} — the *n*-dim simply connected Riemannian manifolds with sectional curvature $\equiv H$.

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Model Spaces

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Ricci Curvatur Comparison Geometry

Laplace Comparsion Proof of the Laplacian Comparison Of Laplacian Comparison Volume Comparison Two Points Distance Second Derivative Comparison First Neumann Eigenvalue Comparison M_{H}^{n} — the *n*-dim simply connected Riemannian manifolds with sectional curvature $\equiv H$.

After scaling, take H = -1, 0, 1. Model spaces are $\mathbb{H}^n, \mathbb{R}^n, \mathbb{S}^n$.

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First Neuman Eigenvalue Comparison

For M^n with curvature $\geq H$, compare M^n with M_H^n .

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Comparison Geometry

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First Neuman Eigenvalue Comparison

For M^n with curvature $\geq H$, compare M^n with M_H^n .

General principle: Bigger curvature ~> Smaller size

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Comparison Geometry

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Laplace Comparsion Proof of the Laplacian Comparison of Laplacian Comparison Volume Comparison Two Points Distance Second Derivative Comparison First Neumann Eigenvalue Comparison For M^n with curvature $\geq H$, compare M^n with M_H^n .

General principle: Bigger curvature ~> Smaller size

Very successful with sectional curvature: Rauch (1951) and Toponogov (1959) comparison theorem

Hessian Comparison for Sectional Curvature

Comparison Results for Ricci Curvature I Comparison Geometry

Given $u \in C^{\infty}(M)$, the Hessian of u is a symmetric 2-tensor:

Hess $u(X, Y) = \langle \nabla_X \nabla u, Y \rangle$.

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Hessian Comparison for Sectional Curvature

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Ricci Curvatur Comparison Geometry

Laplace Comparsion Proof of the Laplacian Comparison Applications of Laplacian Comparison Volume Comparison Two Points Distance Second Derivative Comparison First Neumann Eigenvalue Comparison Given $u \in C^{\infty}(M)$, the Hessian of u is a symmetric 2-tensor:

Hess $u(X, Y) = \langle \nabla_X \nabla u, Y \rangle$.

If $K_M \ge H$, $x \in M$, r(y) = d(x, y), the distance function from x, then the second variation formula and index lemma gives

 $\mathsf{Hess}\,r(e,e) \leq \mathsf{Hess}_H\,\bar{r}(\bar{e},\bar{e})$

for $e \in T_y M$ with |e| = 1 and $e \perp \nabla r(y)$. (same for model)

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Hessian Comparison for Sectional Curvature

Comparison Results for Ricci Curvature I Guofang Wei Introduction Ricci Curvature Comparison Geometry

Laplace Comparsion Proof of the Laplacian Comparison Applications of Laplacian Comparison Volume Comparison Two Points Distance Second Derivative Comparison First Neumann Eigenvalue Comparison Given $u \in C^{\infty}(M)$, the Hessian of u is a symmetric 2-tensor:

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for $e \in T_y M$ with |e| = 1 and $e \perp \nabla r(y)$. (same for model)

Hessian Comparison is an infinitesinal version of Rauch Comparion.

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Comparison Results for Curvature I Laplace Comparsion

Given
$$M^n$$
 with $\operatorname{Ric} \geq (n-1)H$, $x \in M$, $r(\cdot) = d(x, \cdot)$, then

 $\Delta r \leq \Delta_H r.$

Comparison Results for Curvature I Laplace Comparsion

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This is a very fundamental comparison result!

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Comparison Results for Ricci Curvature I Laplace Comparsion

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It characterizes Ricci curvature lower bound:

 $\operatorname{Ric} \geq (n-1)H \iff \Delta r \leq \Delta_H r$

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Comparison Results for Ricci Curvature I Laplace Comparsion

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Used in Cheeger-Gromoll's splitting theorem (1971).

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Comparison Results for Ricci Curvature I

For smooth function u on (M^n, g) ,

 $\frac{1}{2}\Delta |\nabla u|^2 = |\text{Hess } u|^2 + \langle \nabla u, \nabla (\Delta u) \rangle + \text{Ric}(\nabla u, \nabla u).$

Application of Laplacia Comparison Volume Comparison Two Points Distance Second Derivative Comparison First Neuma Eigenvalue Comparison

Laplace Comparsion

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Comparison Results for Ricci Curvature I

Laplace Comparsion

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One can apply it to harmonic function: $\Delta u = 0$,

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Comparison Results for Ricci Curvature I

Laplace Comparsion

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One can apply it to harmonic function: $\Delta u = 0$, distance function: $|\nabla r| = 1$,

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Eigenvalue Comparison

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For smooth function u on (M^n, g) ,

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If u is harmonic and $|\nabla u| = 1$, and Ric ≥ 0 , then Hess u = 0. i.e. ∇u is parallel.

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Comparison Results for Ricci Curvature I

Laplace Comparison

For smooth function u on (M^n, g) ,

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One can apply it to harmonic function: $\Delta u = 0$, distance function: $|\nabla r| = 1$, eigenfunction: $\Delta u = \lambda u$

If u is harmonic and $|\nabla u| = 1$, and Ric ≥ 0 , then Hess u = 0. i.e. ∇u is parallel. This is the starting point of Cheeger-Gromoll's splitting theorem.

Bochner Inequality

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Derivative

First Neuman Eigenvalue Comparison Using the Cauchy-Schwarz inequality $|\text{Hess } u|^2 \ge \frac{(\Delta u)^2}{n}$, if Ric $\ge (n-1)H$,

$$\frac{1}{2}\Delta |\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla (\Delta u) \rangle + (n-1)H|\nabla u|^2$$

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Bochner Inequality

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This characterizes Ricci curvature lower bound.

Bochner Inequality

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Laplace Comparsion Proof of the Laplacian Comparison Applications of Laplacian Comparison Volume Comparison Two Points Distance Second Derivative Comparison First Neuman Eigenvalue Using the Cauchy-Schwarz inequality $|\text{Hess } u|^2 \ge \frac{(\Delta u)^2}{n}$, if Ric $\ge (n-1)H$,

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This characterizes Ricci curvature lower bound.

 $\operatorname{Ric}_M \geq (n-1)H$

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The Bochner inequality holds for all $u \in C^3(M)$

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Proof of the Laplacian Comparison

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First Neuman Eigenvalue Comparison

Recall the Bochner formula

$$\frac{1}{2}\Delta |\nabla u|^2 = |\text{Hess } u|^2 + \langle \nabla u, \nabla (\Delta u) \rangle + \text{Ric}(\nabla u, \nabla u)$$

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Proof of the Laplacian Comparison

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First Neumanı Eigenvalue Comparison

Recall the Bochner formula

$$\frac{1}{2}\Delta |\nabla u|^2 = |\text{Hess } u|^2 + \langle \nabla u, \nabla (\Delta u) \rangle + \text{Ric}(\nabla u, \nabla u)$$

Let u = r, the distance function, we have

$$0 = |\mathsf{Hess}\,r|^2 + (\Delta r)' + \mathsf{Ric}(\partial r, \partial r).$$

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Proof of the Laplacian Comparison

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Proof of the Laplacian Comparison

Recall the Bochner formula

$$\frac{1}{2}\Delta |\nabla u|^2 = |\text{Hess } u|^2 + \langle \nabla u, \nabla (\Delta u) \rangle + \text{Ric}(\nabla u, \nabla u).$$

Let u = r, the distance function, we have

$$0 = |\mathsf{Hess}\,r|^2 + (\Delta r)' + \mathsf{Ric}(\partial r, \partial r).$$

For the model space, we have

$$0 = \frac{(\Delta_H r)^2}{n-1} + (\Delta_H r)' + (n-1)H.$$

For M^n with $\operatorname{Ric}_M \geq (n-1)H$, we have the Riccati inequality

$$0 \geq \frac{(\Delta r)^2}{n-1} + (\Delta r)' + (n-1)H.$$

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Comparison Results for Ricci We have

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$$(\Delta r - \Delta_H r)' \leq -\frac{1}{n-1} \left((\Delta r)^2 - (\Delta_H r)^2 \right).$$

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Ricci

Comparison

$$(\Delta r - \Delta_H r)' \leq -\frac{1}{n-1} \left((\Delta r)^2 - (\Delta_H r)^2 \right)$$

Laplacian Comparison

Let $sn_H(r)$ be the solution to

 $\operatorname{sn}''_H + H \operatorname{sn}_H = 0$

such that $\operatorname{sn}_H(0) = 0$ and $\operatorname{sn}'_H(0) = 1$. Then

$$\Delta_H r = (n-1) \frac{\mathrm{sn}'_H}{\mathrm{sn}_H}$$

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Proof of the Laplacian Comparison

> First Neuman Eigenvalue Comparison

We compute that

 $\left(\operatorname{sn}_{H}^{2}(\Delta r - \Delta_{H}r)\right)^{\prime}$ $= 2 \mathrm{sn}'_{H} \mathrm{sn}_{H} (\Delta r - \Delta_{H} r) + s n_{H}^{2} (\Delta r - \Delta_{H} r)'$

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We compute that

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 $\begin{aligned} \left(\operatorname{sn}_{H}^{2} (\Delta r - \Delta_{H} r) \right)' \\ &= 2 \operatorname{sn}_{H}' \operatorname{sn}_{H} (\Delta r - \Delta_{H} r) + s n_{H}^{2} (\Delta r - \Delta_{H} r)' \\ &\leq \frac{2}{n-1} \operatorname{sn}_{H}^{2} \Delta_{H} r (\Delta r - \Delta_{H} r) - \frac{1}{n-1} \operatorname{sn}_{H}^{2} \left((\Delta r)^{2} - (\Delta_{H} r)^{2} \right) \end{aligned}$

We compute that

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$$\begin{aligned} \left(\operatorname{sn}_{H}^{2} (\Delta r - \Delta_{H} r) \right)' &= 2 \operatorname{sn}_{H}' \operatorname{sn}_{H} (\Delta r - \Delta_{H} r) + s n_{H}^{2} (\Delta r - \Delta_{H} r)' \\ &\leq \frac{2}{n-1} \operatorname{sn}_{H}^{2} \Delta_{H} r (\Delta r - \Delta_{H} r) - \frac{1}{n-1} \operatorname{sn}_{H}^{2} \left((\Delta r)^{2} - (\Delta_{H} r)^{2} \right) \\ &= -\frac{\operatorname{sn}_{H}^{2}}{n-1} \left(\Delta r - \Delta_{H} r \right)^{2} \leq 0 \end{aligned}$$

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We compute that

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$$\begin{aligned} \left(\operatorname{sn}_{H}^{2} (\Delta r - \Delta_{H} r) \right)' \\ &= 2 \operatorname{sn}_{H}' \operatorname{sn}_{H} (\Delta r - \Delta_{H} r) + s n_{H}^{2} (\Delta r - \Delta_{H} r)' \\ &\leq \frac{2}{n-1} \operatorname{sn}_{H}^{2} \Delta_{H} r (\Delta r - \Delta_{H} r) - \frac{1}{n-1} \operatorname{sn}_{H}^{2} \left((\Delta r)^{2} - (\Delta_{H} r)^{2} \right) \\ &= -\frac{\operatorname{sn}_{H}^{2}}{n-1} \left(\Delta r - \Delta_{H} r \right)^{2} \leq 0 \end{aligned}$$

Since $\lim_{r\to 0} \operatorname{sn}^2_H(\Delta r - \Delta_H r) = 0$, integrating from 0 to s yields $\operatorname{sn}^2_H(s)(\Delta r(s) - \Delta_H r(s)) \le 0,$

The Laplacian Comparison!

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> First Neumani Eigenvalue Comparison

Immediately gives the Bonnet-Myers (1941) theorem: If $\operatorname{Ric}_M \ge (n-1)H > 0$, then $\operatorname{Diam}_M \le \operatorname{Diam}(\mathbb{S}_H^n) = \frac{\pi}{\sqrt{H}}$;

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Comparison Results for Ricci Curvature I

ntroduction Ricci Curvature Comparison

Laplace Comparsion Proof of the Laplacian Comparison Applications of Laplacian Comparison Volume Comparison Two Points Distance Second Derivative Comparison First Neumann First Neumann Immediately gives the Bonnet-Myers (1941) theorem: If $\operatorname{Ric}_M \ge (n-1)H > 0$, then $\operatorname{Diam}_M \le \operatorname{Diam}(\mathbb{S}_H^n) = \frac{\pi}{\sqrt{H}}$;

Also gives Dirichlet Eigenvalue, Heat Kernel Comparisons: If $\operatorname{Ric}_{M} \geq (n-1)H$, then (Cheng, 1975) $\lambda_{1}(B(x, R)) \leq \lambda_{1}(B_{H}(R))$. (Cheeger-Yau, 1981) $H(x, y, t) \geq H_{H}(\overline{x, y}, t)$.

Comparison of Volume Elements

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aplace

Proof of the

Applications of Laplacian Comparison Volume Comparison Two Points Distance Second Derivative Comparison First Neumann Eigenvalue Suppose M^n has $Ric_M \ge (n-1)H$. Wirte the volume element $dvol = \mathcal{A}(r, \theta) drd\theta_{n-1}$ in the geodesic polar coordinate at q and same for the model space $dvol_H = \mathcal{A}_H(r) drd\theta_{n-1}$. Then



is nonincreasing along any minimal geodesic segment from q.

This follows from the following lemma and the Laplacian comparison.

Lemma

Theorem

 $\frac{\mathcal{A}'}{4}(r,\theta) = \Delta r.$

Bishop-Gromov Volume Comparison

Comparison Results for Ricci Curvature I Volume Comparison

Given M^n with Ric $\geq (n-1)H$, $x \in M$. Let Vol_H(B(r)) be the volume of *r*-ball in the model space M_H^n , then

 $\frac{\text{Vol}(B(x,r))}{\text{Vol}_{H}(B(r))}$ is nonincreasing in r.

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Bishop-Gromov Volume Comparison

Results for Curvature I Volume Comparison

Comparison

Given M^n with Ric $\geq (n-1)H$, $x \in M$. Let Vol_H(B(r)) be the volume of *r*-ball in the model space M_H^n , then

 $\frac{\text{Vol}(B(x,r))}{\text{Vol}_{H}(B(r))}$ is nonincreasing in r.

In particular,

 $\begin{aligned} & \operatorname{Vol}\left(B(p,R)\right) \leq \operatorname{Vol}_{H}(B(R)) & \text{for all } R > 0, \\ & \frac{\operatorname{Vol}\left(B(p,r)\right)}{\operatorname{Vol}\left(B(p,R)\right)} \geq \frac{\operatorname{Vol}_{H}(B(r))}{\operatorname{Vol}_{H}(B(R))} & \text{for all } 0 < r \leq R. \end{aligned}$

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Bishop-Gromov Volume Comparison

Comparison Results for Ricci Curvature I Volume Comparison

Given M^n with Ric $\geq (n-1)H$, $x \in M$. Let Vol_H(B(r)) be the volume of r-ball in the model space M_H^n , then

 $\frac{\operatorname{Vol}(B(x,r))}{\operatorname{Vol}_{H}(B(r))}$ is nonincreasing in r.

In particular,

 $\begin{aligned} & \operatorname{Vol}\left(B(p,R)\right) \leq \operatorname{Vol}_{H}(B(R)) & \text{for all } R > 0, \\ & \frac{\operatorname{Vol}\left(B(p,r)\right)}{\operatorname{Vol}\left(B(p,R)\right)} \geq \frac{\operatorname{Vol}_{H}(B(r))}{\operatorname{Vol}_{H}(B(R))} & \text{for all } 0 < r \leq R. \end{aligned}$

Moreover equality holds if and only if B(p, R) is isometric to $B_H(R)$.

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Two Point Distance Second Derivative

> First Neuman Eigenvalue Comparison

• Milnor's result on the growth of π_1 for manifolds with Ric \geq 0 (1968)

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- Milnor's result on the growth of π_1 for manifolds with Ric \geq 0 (1968)
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Eigenvalue Comparison

- Milnor's result on the growth of π_1 for manifolds with Ric \geq 0 (1968)
- Gromov's first Betti number estimate, precompactness theorem
- Cheng's maximal diameter theorem

Comparison Results for Ricci Curvature I Volume Comparison

■ Milnor's result on the growth of π₁ for manifolds with Ric ≥ 0 (1968)
 ■ Gromov's first Betti number estimate, precompactness

theorem

- Cheng's maximal diameter theorem
- Volume growth of noncompact manifolds with $Ric \ge 0$.

Two Points Distance Second Derivative Comparison

Comparison Results for Ricci Curvature I Two Points Distance Second Derivative Comparison

The Laplacian comparison is for one point distance function in r(x, y), we fix x. If we let both vary, one has

Lemma (Andrews and Clutterbuck 2013)

 M^n with $Ric_M \ge (n-1)H$, then

$$\sum_{i=1}^{n-1} \nabla_{e_i,e_i}^2 r(x,y) \leq \sum_{i=1}^{n-1} \bar{\nabla}_{e_i,e_i}^2 r(x,y) \\ = -2(n-1)T_H(\frac{r(x,y)}{2})$$

where $e_i \perp \nabla r$ are orthonormal and parallel, $T_H = H \frac{sn_H}{(sn_H)'}$.

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Laplacian Comparison for Radial Functions

Comparison Results for Ricci Curvature I In geodesic polar coordinate, we have

$$\Delta = ilde{\Delta} + (\Delta r) rac{\partial}{\partial r} + rac{\partial^2}{\partial r^2},$$

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Two Points Distance Second Derivative Comparison First Neuman Eigenvalue where $\tilde{\Delta}$ is the induced Laplacian on the sphere. Therefore

Theorem (Global Laplacian Comparison)

If $Ric_{M^n} \ge (n-1)H$, in weak (barrier, viscosity or distributions) sense , we have

 $\begin{array}{rcl} \Delta \varphi(r) & \leq & \Delta_H \varphi(r) & (\textit{if } \varphi' \geq 0), \\ \Delta \varphi(r) & \geq & \Delta_H \varphi(r) & (\textit{if } \varphi' \leq 0). \end{array}$

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Laplacian Comparison for Radial Functions

Comparison Results for Ricci Curvature I In geodesic polar coordinate, we have

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barrier subsolutions are viscosity subsolutions; viscosity subsolution iff distribution subsolution

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First Neumanr Eigenvalue Comparison

Let
$$v(x, y) = 2\varphi(\frac{r(x, y)}{2})$$
, with $\varphi' \ge 0$. If $\operatorname{Ric}_{M^n} \ge (n-1)H$, then

$$\sum_{i=1}^{m-1} \nabla_{e_i,e_i}^2 v(x,y) \le -2(n-1)\varphi' T_H(\frac{r(x,y)}{2})$$

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, with $\varphi' \ge 0$. If $\operatorname{Ric}_{M^n} \ge (n-1)H$,
hen

$$\sum_{i=1} \nabla_{e_i,e_i}^2 v(x,y) \leq -2(n-1)\varphi' T_H(\frac{r(x,y)}{2})$$

This is very useful in estimating the modulus of continuity or oscillations!

Application – First Neumann Eigenvalue Comparison

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Theorem

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Comparison Two Points Distance Second Derivative Comparison First Neumann Eigenvalue Comparison Let M^n be a (closed) compact manifold with diameter D, and $Ric_M \ge (n-1)H$. Then the first non-trivial eigenvalue

 $\lambda_1(M,g) \geq \overline{\lambda}(n,H,D),$

where $\overline{\lambda}(n, H, D)$ is the first eigenvalue of the operator

 $L_{H,D} = \varphi''(s) - (n-1)\varphi'T_H(s)$

on the interval $\left[-\frac{D}{2}, \frac{D}{2}\right]$ with Neumann boundary $\varphi'(\pm \frac{D}{2}) = 0$.

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Remark

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First Neumann Eigenvalue Comparison • When H > 0, it gives the Lichnerowicz (1958) first eigenvalue comparison: $\lambda_1 \ge \lambda_1(M_H^n) = nH$.

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Remark

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First Neumann Eigenvalue Comparison

- When H > 0, it gives the Lichnerowicz (1958) first eigenvalue comparison: $\lambda_1 \ge \lambda_1(M_H^n) = nH$.
- When H = 0, it gives Zhong-Yang (1984) estimate: $\lambda_1 \geq \frac{\pi}{D^2}$.

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Remark

Comparison Results for Ricci Curvature I First Neumann Eigenvalue Comparison

- When H > 0, it gives the Lichnerowicz (1958) first eigenvalue comparison: $\lambda_1 \ge \lambda_1(M_H^n) = nH$.
- When H = 0, it gives Zhong-Yang (1984) estimate: $\lambda_1 \ge \frac{\pi}{D^2}$.
- General Case: Mufa Chen and Fengyu Wang (1994) (probabilistic 'coupling method');
 P. Kröger (1998) (using gradient estimate);
 Andrews and Clutterbuck (2013) (modulus of continuity for heat equation);
 Zhang-Wang (2015) (elliptic proof)

Proof of the Eigenvalue Comparison

Comparison Results for Ricci Curvature I

Let $\varphi, \bar{\varphi}$ be the first eigenfunction of the Laplace of M^n and $L_{H,D}$ respectively.

Let

Ricci Curvatı Comparison Geometry

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First Neumann Eigenvalue Comparison $Q(x,y) = \frac{\varphi(y) - \varphi(x)}{\bar{\varphi}\left(\frac{r(x,y)}{2}\right)}$

be the quotient of the oscillations.

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Proof of the Eigenvalue Comparison

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Let

Ricci Curvati Comparison Geometry

Laplace Comparsion Proof of the Laplacian Comparison Applications of Laplacian Comparison Two Points Distance Second Derivative Comparison First Neumann Eigenvalue Comparison Let $\varphi, \bar{\varphi}$ be the first eigenfunction of the Laplace of M^n and $L_{H,D}$ respectively.

 $Q(x,y) = \frac{\varphi(y) - \varphi(x)}{\bar{\varphi}\left(\frac{r(x,y)}{2}\right)}$

be the quotient of the oscillations. Assume the maximum of Q is attained at (x_0, y_0) and $x_0 \neq y_0$. Denote $s_0 = \frac{r(x_0, y_0)}{2}$. At the maximum point (x_0, y_0) , we have

$$\begin{array}{rcl} \overline{\gamma}^2_{e_i,e_i}Q & = & \displaystyle \frac{1}{\bar{\varphi}(s_0)}\left(\nabla^2_{e_i,e_i}(\varphi(y_0)-\varphi(x_0))-Q(x_0,y_0)\nabla^2_{e_i,e_i}\bar{\varphi}(s_0)\right.\\ & \leq & \displaystyle 0 \end{array}$$

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First Neumann Eigenvalue Comparison Since $\bar{\varphi}(s_0) > 0$, $Q(x_0, y_0) > 0$, sum over $i = 1, \dots, n-1$ and using the comparison for $\nabla^2_{e_i, e_i} \bar{\varphi}(s_0)$, we have

$$\sum_{i=1}^{n-1} \nabla_{e_i,e_i}^2(\varphi(y_0) - \varphi(x_0)) \leq Q(x_0,y_0) \left[-(n-1)\bar{\varphi}'(s_0) T_H(s_0) \right]$$

Add the radial direction gives

$$\Delta \varphi(y_0) - \Delta \varphi(x_0) \le Q(x_0, y_0) \left[\overline{\varphi}''(s_0) - (n-1)\overline{\varphi}'(s_0) T_H(s_0) \right]$$

i.e.

$$\begin{array}{rcl} \lambda_1(M,g)(\varphi(y_0)-\varphi(x_0)) &\leq & Q(x_0,y_0)\bar{\lambda}(n,H,D)\bar{\varphi}(s_0).\\ &\lambda_1(M,g) &\leq & \bar{\lambda}(n,H,D) \end{array}$$

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Volume Comparison Two Points Distance Second Derivative Comparison First Neumann Eigenvalue Comparison When the maximum of Q is attained at (x_0, y_0) and $x_0 = y_0$, a limiting process reduce to above. Note that as $y \to x$,

$$Q(x,\gamma'(0))=rac{2\langle
ablaarphi(x),\gamma'(0)
angle}{ar{arphi}'(0)}$$

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Lecture II

Generalizations to Bakry-Emery and Integral Ricci Curvaure

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Lecture II

Generalizations to Bakry-Emery and Integral Ricci Curvaure

Thank you