# Comparison Results for Ricci Curvature I

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# Ricci Tensor





Given a Riemannian manifold $(M^n, g)$, the Ricci tensor

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} \langle R(X, e_i)e_i, Y\rangle$$

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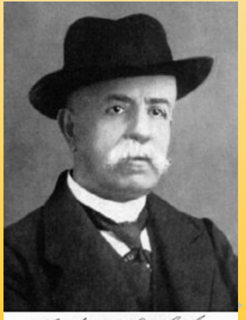
Italian mathematician: Gregorio Ricci-Curbastro (1853-1925)



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Ricci curvature is in between sectional curvature and scalar curvature, measures the deviation of the volume element from the Euclidean one.

- Vacuum Einstein equation

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- Related to Optimal Transportation

# Model Spaces

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After scaling, take $H = -1, 0, 1$.
Model spaces are $\mathbb{H}^n, \mathbb{R}^n, \mathbb{S}^n$.

# Comparison Geometry

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Very successful with sectional curvature: Rauch (1951) and Toponogov (1959) comparison theorem

# Hessian Comparison for Sectional Curvature

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If $K_M \geq H$, $x \in M$, $r(y) = d(x, y)$, the distance function from $x$, then the second variation formula and index lemma gives

$$\text{Hess}\, r(e, e) \leq \text{Hess}_H\, \bar{r}(\bar{e}, \bar{e})$$

for $e \in T_y M$ with $|e| = 1$ and $e \perp \nabla r(y)$. (same for model)

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Hessian Comparison is an infinitesinal version of Rauch Comparion.

# Laplacian Comparison

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Used in Cheeger-Gromoll's splitting theorem (1971).

# A main tool for Ricci curvature — Bochner formula





For smooth function $u$ on $(M^n, g)$,

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u).$$

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i.e. $\nabla u$ is parallel.
This is the starting point of Cheeger-Gromoll's splitting theorem.

# Bochner Inequality



Using the Cauchy-Schwarz inequality $|\mathrm{Hess}\, u|^2 \geq \frac{(\Delta u)^2}{n}$, if $\mathrm{Ric} \geq (n-1)H$,

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This characterizes Ricci curvature lower bound.

$$\text{Ric}_M \geq (n-1)H$$

$$\Updownarrow$$

The Bochner inequality holds for all $u \in C^3(M)$

# Proof of the Laplacian Comparison

Recall the Bochner formula

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Let $u = r$, the distance function, we have

$$0 = |\mathrm{Hess}\, r|^2 + (\Delta r)' + \mathrm{Ric}(\partial r, \partial r).$$

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For the model space, we have

$$0 = \frac{(\Delta_H r)^2}{n-1} + (\Delta_H r)' + (n-1)H.$$

For $M^n$ with $\mathrm{Ric}_M \geq (n-1)H$, we have the Riccati inequality

$$0 \geq \frac{(\Delta r)^2}{n-1} + (\Delta r)' + (n-1)H.$$

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Let $\mathrm{sn}_H(r)$ be the solution to

$$\mathrm{sn}_H'' + H\,\mathrm{sn}_H = 0$$

such that $\mathrm{sn}_H(0) = 0$ and $\mathrm{sn}_H'(0) = 1$. Then

$$\Delta_H r = (n-1)\frac{\mathrm{sn}_H'}{\mathrm{sn}_H}.$$

We compute that

$$
\begin{aligned}
&\left(\mathrm{sn}_H^2(\Delta r - \Delta_H r)\right)' \\
&\quad = \; 2\mathrm{sn}'_H \mathrm{sn}_H(\Delta r - \Delta_H r) + sn_H^2(\Delta r - \Delta_H r)'
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&\quad \le \frac{2}{n-1}\mathrm{sn}_H^2\Delta_H r(\Delta r - \Delta_H r) - \frac{1}{n-1}\mathrm{sn}_H^2\left((\Delta r)^2 - (\Delta_H r)^2\right)
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Since $\lim_{r\to 0}\mathrm{sn}_H^2(\Delta r - \Delta_H r) = 0$, integrating from 0 to s yields

$$\mathrm{sn}_H^2(s)(\Delta r(s) - \Delta_H r(s)) \le 0,$$

The Laplacian Comparison!

# Some Applications





Immediately gives the Bonnet-Myers (1941) theorem:
If $\mathrm{Ric}_M \geq (n-1)H > 0$, then $\mathrm{Diam}_M \leq \mathrm{Diam}\,(\mathbb{S}^n_H) \;= \frac{\pi}{\sqrt{H}}$;

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Immediately gives the Bonnet-Myers (1941) theorem:
If $\text{Ric}_M \geq (n-1)H > 0$, then $\text{Diam}_M \leq \text{Diam}\,(\mathbb{S}^n_H) \;= \frac{\pi}{\sqrt{H}}$;

Also gives Dirichlet Eigenvalue, Heat Kernel Comparisons:
If $\text{Ric}_M \geq (n-1)H$, then
(Cheng, 1975) $\lambda_1(B(x,R)) \leq \lambda_1(B_H(R))$.
(Cheeger-Yau, 1981) $H(x,y,t) \geq H_H(\overline{x,y},t)$.

# Comparison of Volume Elements



## Theorem

*Suppose $M^n$ has $Ric_M \geq (n-1)H$. Wirte the volume element $dvol = \mathcal{A}(r,\theta)\, dr d\theta_{n-1}$ in the geodesic polar coordinate at $q$ and same for the model space $dvol_H = \mathcal{A}_H(r)\, dr d\theta_{n-1}$. Then*

$$\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}$$

*is nonincreasing along any minimal geodesic segment from $q$.*

This follows from the following lemma and the Laplacian comparison.

## Lemma

$$\frac{\mathcal{A}'}{\mathcal{A}}(r,\theta) = \Delta r.$$

# Bishop-Gromov Volume Comparison





Given $M^n$ with $\mathrm{Ric} \geq (n-1)H$, $x \in M$. Let $\mathrm{Vol}_H(B(r))$ be the volume of $r$–ball in the model space $M^n_H$, then

$$\frac{\mathrm{Vol}\,(B(x,r))}{\mathrm{Vol}_H(B(r))} \text{ is nonincreasing in } r.$$

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In particular,

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Moreover equality holds if and only if $B(p,R)$ is isometric to $B_H(R)$.

# Some Applications

- Milnor's result on the growth of $\pi_1$ for manifolds with $\text{Ric} \geq 0$ (1968)

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- Gromov's first Betti number estimate, precompactness theorem
- Cheng's maximal diameter theorem
- Volume growth of noncompact manifolds with $\mathrm{Ric} \geq 0$.

# Two Points Distance Second Derivative Comparison





The Laplacian comparison is for one point distance function — in $r(x, y)$, we fix $x$.
If we let both vary, one has

**Lemma (Andrews and Clutterbuck 2013)**

*$M^n$ with $Ric_M \geq (n-1)H$, then*

$$\sum_{i=1}^{n-1} \nabla^2_{e_i, e_i} r(x,y) \leq \sum_{i=1}^{n-1} \bar{\nabla}^2_{e_i, e_i} r(x,y) = -2(n-1) T_H\left(\frac{r(x,y)}{2}\right)$$

*where $e_i \perp \nabla r$ are orthonormal and parallel, $T_H = H \frac{sn_H}{(sn_H)'}$.*

# Laplacian Comparison for Radial Functions

In geodesic polar coordinate, we have

$$\Delta = \tilde{\Delta} + (\Delta r)\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2},$$

where $\tilde{\Delta}$ is the induced Laplacian on the sphere. Therefore

**Theorem (Global Laplacian Comparison)**

*If $Ric_{M^n} \geq (n-1)H$, in weak (barrier, viscosity or distributions) sense , we have*

$$\begin{aligned} \Delta\varphi(r) &\leq \Delta_H\varphi(r) \quad (\text{if } \varphi' \geq 0), \\ \Delta\varphi(r) &\geq \Delta_H\varphi(r) \quad (\text{if } \varphi' \leq 0). \end{aligned}$$

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barrier subsolutions are viscosity subsolutions;
viscosity subsolution iff distribution subsolution

# Two Points Radial Functions Second Derivative Comparison



Let $v(x,y) = 2\varphi(\frac{r(x,y)}{2})$, with $\varphi' \geq 0$. If $\mathrm{Ric}_{M^n} \geq (n-1)H$, then

$$\sum_{i=1}^{n-1} \nabla^2_{e_i,e_i} v(x,y) \leq -2(n-1)\varphi' T_H(\frac{r(x,y)}{2})$$

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This is very useful in estimating the modulus of continuity or oscillations!

# Application – First Neumann Eigenvalue Comparison

**Theorem**

*Let $M^n$ be a (closed) compact manifold with diameter $D$, and $Ric_M \geq (n-1)H$. Then the first non-trivial eigenvalue*

$$\lambda_1(M, g) \geq \bar{\lambda}(n, H, D),$$

*where $\bar{\lambda}(n, H, D)$ is the first eigenvalue of the operator*

$$L_{H,D} = \varphi''(s) - (n-1)\varphi' T_H(s)$$

*on the interval $[-\frac{D}{2}, \frac{D}{2}]$ with Neumann boundary $\varphi'(\pm\frac{D}{2}) = 0$.*

# Remark

- When $H > 0$, it gives the Lichnerowicz (1958) first eigenvalue comparison: $\lambda_1 \geq \lambda_1(M^n_H) = nH$.

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# Remark

- When $H > 0$, it gives the Lichnerowicz (1958) first eigenvalue comparison: $\lambda_1 \geq \lambda_1(M^n_H) = nH$.
- When $H = 0$, it gives Zhong-Yang (1984) estimate: $\lambda_1 \geq \frac{\pi}{D^2}$.
- General Case: Mufa Chen and Fengyu Wang (1994) (probabilistic 'coupling method');
  P. Kröger (1998) (using gradient estimate);
  Andrews and Clutterbuck (2013) (modulus of continuity for heat equation);
  Zhang-Wang (2015) (elliptic proof)

# Proof of the Eigenvalue Comparison

Let $\varphi, \bar{\varphi}$ be the first eigenfuntion of the Laplace of $M^n$ and $L_{H,D}$ respectively.
Let

$$Q(x, y) = \frac{\varphi(y) - \varphi(x)}{\bar{\varphi}\left(\frac{r(x,y)}{2}\right)}$$

be the quotient of the oscillations.

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be the quotient of the oscillations.
Assume the maximum of $Q$ is attained at $(x_0, y_0)$ and $x_0 \neq y_0$.
Denote $s_0 = \frac{r(x_0,y_0)}{2}$.
At the maximum point $(x_0, y_0)$, we have

$$\begin{aligned} \nabla^2_{e_i,e_i} Q &= \frac{1}{\bar{\varphi}(s_0)} \left(\nabla^2_{e_i,e_i}(\varphi(y_0) - \varphi(x_0)) - Q(x_0,y_0) \nabla^2_{e_i,e_i} \bar{\varphi}(s_0)\right) \\ &\leq 0 \end{aligned}$$

Since $\bar{\varphi}(s_0) > 0,\ Q(x_0, y_0) > 0$, sum over $i = 1, \cdots, n-1$ and using the comparison for $\nabla^2_{e_i,e_i}\bar{\varphi}(s_0)$, we have

$$\sum_{i=1}^{n-1} \nabla^2_{e_i,e_i}(\varphi(y_0) - \varphi(x_0)) \le Q(x_0, y_0)\left[-(n-1)\bar{\varphi}'(s_0)T_H(s_0)\right]$$

Add the radial direction gives

$$\Delta\varphi(y_0) - \Delta\varphi(x_0) \le Q(x_0, y_0)\left[\bar{\varphi}''(s_0) - (n-1)\bar{\varphi}'(s_0)T_H(s_0)\right]$$

i.e.

$$\begin{aligned} \lambda_1(M,g)(\varphi(y_0) - \varphi(x_0)) &\le Q(x_0, y_0)\bar{\lambda}(n, H, D)\bar{\varphi}(s_0). \\ \lambda_1(M,g) &\le \bar{\lambda}(n, H, D) \end{aligned}$$

When the maximum of $Q$ is attained at $(x_0, y_0)$ and $x_0 = y_0$, a limiting process reduce to above. Note that as $y \to x$,

$$Q(x, \gamma'(0)) = \frac{2\langle \nabla \varphi(x), \gamma'(0)\rangle}{\bar{\varphi}'(0)}$$

.

# Lecture II

Generalizations to Bakry-Emery and Integral Ricci Curvaure

# Next Lecture





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## Thank you