# RICCI FLAT SPACES AND METRICS WITH SPECIAL OR EXCEPTIONAL HOLONOMY

Overview: searching for Ricci flat metrics & ones w/ exceptional holonomy

Ricci-flat mflds

1975 - Fischer & Wolf   structure thms to uncover more flat mflds
Still mysterious

1978 - Yau   proved the Calabi conjecture. $\Rightarrow$ lots of cmpt Kähler mflds w/ Ricci $= 0$.

think in the context of
Einstein mflds $\longrightarrow$ consider various homogeneous examples
$R = \lambda g$

Nontrivial cmpt Ricci-flat mflds are difficult to study, highly nonsymmetric, etc.

1987   Book by Besse on Einstein mflds, claims how difficult examples are to find.

mid 90's   Joyce found cmpt Ricci flat 7-mflds, also arise from special holonomy

Let's go back in time and discuss Riemannian holonomy

Start looking at parallel transport about closed loops.
Holonomy   which $G \subseteq SO(n)$ arise as the (restricted) holonomy group of a Riem mfld

1920's   Cartan defines & studies locally symmetric $G/H$, $\nabla R \equiv 0$.

If $\nabla R \neq 0$, holonomy groups are much more constrained (see Joyce's text)
Philosophy: The holonomy constrains curvature, conversely the curv determines the holonomy (Ambrose-Singer thm)

1955   Berger   irreducible non locally symmetric Riemannian mflds
"8 families could arise" (one was eventually ruled out)

(i) $SO(n)$ (the generic holonomy)
(ii) $U(n) \subseteq SO(2n) \leftrightarrow$ Kähler
1978 (iii) $SU(n) \subseteq SO(2n) \leftrightarrow$ Calabi-Yau
(iv) $Sp(n) \subseteq SO(4n) \leftrightarrow$ hyper Kähler
(v) $Sp(n)Sp(1)$ Quaternioni Kähler
(vi) $G_2 \subseteq SO(7)$
(vii) $Spin_7 \subseteq SO(8)$
exceptional holonomy 1984, 1989, 1995
(ix) $Spin_9$, ruled out since we need locally symmetric

1st Observation (iii), (iv), (vi), (vii) automatically Ricci flat
slick proof by Robert Bryant.
most pfs done by Bonan, mid 60's.

In Kähler case can think of curvature as 1-1 form.
Ric $\rightsquigarrow$ curvature of anticanonical line bundle.

$$\mathrm{Ric} = 0, \quad \Rightarrow c_1(M) = 0$$
?
This was part of the Calabi conjecture.

$\omega = \omega_0 + i\partial\bar{\partial}\varphi$ solving $Rc = 0$ is equivalent to solving complex Monge Ampere
reduced to a nonlinear problem $\rightarrow$ global reduction of eqn to scalar, elliptic, non linear PDE

Yau's thm is powerful, but it doesn't give a good idea of what the metric solutions look like.

Noncompact examples $\to$ compact $\Rightarrow$ no symmetries (unless they split)

noncompact $\to$

1979 Calabi non linear ODE's max symmetry $\to$ codim 1

$\Downarrow$ explicitly solvable $G \circlearrowright M$ codim orbits of $G = 1$

For work in $G_2$ world we won't necessarily have nice structure or strategies as above

Simplest Ricci Flat compact mfld is K3 surface.

Kummer Construction : $T / \Gamma$ $\leftarrow$ torus, $\leftarrow$ finite gp of symmetries

Hyper Kähler singular orbifold $\to$ resolve the singularities

with proper choice of resolution (metric on resolution)

can create $(X, \omega)$ which is almost Ricci Flat

$\uparrow$ Kahler metric $\downarrow$ then use perturbation in an IFT-type way.

Q: What is a $G_2$ manifold?

Simple compact Lie groups $\to$ Killing wanted to show only 2 families $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, then found Symplectic and eventually $G_2$

invented "the root system"

Cartan eventually reworked this & found a 7-dim'l irred representation.

To see relation consider the Octonians.

Nonassociativity is not a problem in this case.

Quaternions $\mathbb{H}$ $\mathbb{R}^3 \approx \operatorname{Im}\mathbb{H}$.

$u, v \in \operatorname{Im}(\mathbb{H})$

$u \cdot v = \langle u, v\rangle\, 1 + \operatorname{Im}(uv)$ $\swarrow u \times v$

Do something analogous on octonians. $\mathbb{R}^7 \cong \operatorname{Im}\mathbb{O}$, $u \times v = \operatorname{Im}(uv)$, for $u, v \in \mathbb{R}^7$.

$\langle u \times v, w\rangle = \varphi(u, v, w)$ 3-form

$G_2$-mfld is the automorphisms of the octonians preserves $\varphi$.

In fact, $G_2 = \{ A \in GL_7(\mathbb{R}) : A^*\varphi = \varphi \}$

$\dim G_2 = 14$, $\dim \Lambda^3 \mathbb{R}^7 = 35$, so $49 - 14 = 35$

the orbit of $\varphi$ under $GL_7\mathbb{R}$ is open

You get an indefinite metric w/ signature $(3,4)$. Idea: don't work directly w/ metric, work w/ 3-form.

We want to recast the holonomy problem:

$G_2$ holonomy metric $\Rightarrow$ $\exists$ parallel 3-form $\varphi$ modelled on O.

$\operatorname{Hol} \subseteq G_2$. $\Leftarrow$

$G_2$ structure on a 7-mfld.

smoothly varying choice of 3-form $\varphi$ ptwise in this given open set.

$\varphi$ exists $\Leftrightarrow$ mfld is spin. Since $G_2 \subseteq SO(7)$ then $\varphi$ determines a metric (but in a nonlinear way)

Q: when is $\operatorname{Hol}_{g_\varphi} \subseteq G_2$?

TFAE on a cpt 7-mfld

$(M, \varphi, g_\varphi)$ a $G_2$ structure

(1) $\mathrm{Hol}_{g_\varphi} \subseteq G_2$

(2) $\nabla \varphi = 0$

(3) $\boxed{d\varphi = d^* \varphi = 0}$ ← exploit this to find compt $G_2$ mfld by mimicking Kummer strategy