ASPECTS OF EINSTEIN METRICS ON 4 MANIFOLDS

(M^n, g) Riemannian manifold

$Ric_g = \lambda g$ most natural eqn imposed on metric

arose via Einstein's work in General Relativity. Lorentzian
interested in cases w/ splitting

$\underset{\text{space}}{\Sigma} \times \underset{\text{time}}{\mathbb{R}} \simeq M$ Want Existence & Uniqueness, std question for both geometry & PDE

Want <u>nice</u> local coordinates which simplify PDE: Einstein equation in harmonic coordinates

$\Delta_g x^i = 0.$ $\quad \underbrace{\Delta_g g_{\alpha\beta}}_{\text{Leading order} \to \text{elliptic}} + Q_{\alpha\beta}(g, \partial g) = -2 Ric_{\alpha\beta}$

System of 2^{nd} order quasi linear eqns

This is equivalent to the Bochner formula

$\Rightarrow -2\lambda g_{\alpha\beta}$

For Lorentzian metric have similarly $\Box_g g_{\alpha\beta} + \cdots$

In the Lorentzian (GR) case, the einstein equations are well posed. (1952 Choquet Bruhat)

$(\Sigma, g, \dot{g}) \to \exists!$ solution on $M = \Sigma \times I$ Main Q: Long time behavior? Singularity formation?

arb. mfld $\quad \dot{g}$ is K, 2^{nd} fundamental form

Closely tied to Ricci flow

Let's turn back to Riemannian setting

<u>Uniqueness</u> ... more generally, structure of solutions space.

$\mathbb{E}$ = set of all solutions of Einstein equations on $M \subseteq Met^{\infty}(M) \subseteq Met^{m,\alpha}(M)$ ← Better to work with

(M compact closed) normalize vol M = 1

Frechet space, not Banach

$\mathcal{E} := \mathbb{E}/Diff(M)$ = moduli space of solutions

(Compare to YM moduli space, self dual instanton equations.)

Donaldson's Thm

Basic Facts

- each component of $\mathcal{E}$ is a finite dimensional real analytic variety (Proven by Koizo)
  however dim = ?
- λ is constant on each component (Einstein metrics are critical pts of E-H action $\int_M R_g \, dv_g$).

Open Q: Are component of $\mathcal{E}$ manifolds or orbifolds.

n = 2
n = 3] completely known & understood.

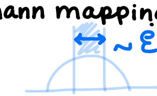
n = 4

n > 4: much different behavior. Thm (Wang, Ziller) $\exists \infty$-many components to $\mathcal{E}_+$ on $S^3 \times S^2$

dim 2 $\mathcal{E}$ = Riemannian moduli space ~ Teichmüller Theory

① • S^2, $\mathcal{E} = \mathcal{E}_+ = pt$ ~ Riemann mapping theorem

② • T^2 $\mathcal{E}_0 = \mathbb{H}^2 / SU(2,2)$



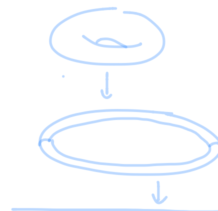
Weil Peterson metric = L^2 metric

$T_g(Met^\infty(M)) = S^2(M)$ $\qquad h = \frac{d}{dt}(g + th)|_{t=0}.$

$$\langle h, k\rangle = \int_M \langle h, M\rangle_g(x)\, dV_g^x$$

W-P metric is L^2 metric restricted to $\mathcal{E}$, the 2-D submanifold.

It's noncompact and geometr. associated to collapse.



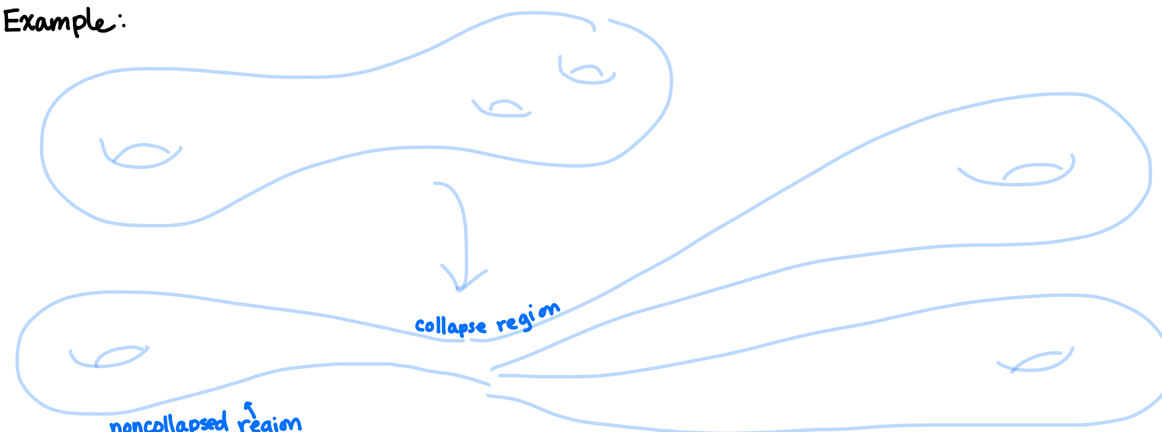
③ • Σ_g, $g \geq 2$ surface $\qquad \mathcal{E}_- = \dfrac{\mathbb{R}^{6g-6}}{\Gamma}$ ← mapping class group

Consider completion wrt WP = L^2 metric $\qquad \bar{\mathcal{E}}_-$ is compact $\qquad$ Note $\text{diam}_{WP}\, \mathcal{E}_- < \infty$

No infinite geodesics

boundary is hyperbolic cusps.

Example:



dim 3 $\mathcal{E}_+, \mathcal{E}_- = pts$ (Mostow rigidity thm) Calabi Weil.

$\mathcal{E}_0$ = moduli space of flat metrics on M^3.

dim4 Picture is very similar to dim 2.

M^4, $\mathcal{E}_\lambda$ = component of $\mathcal{E}$, $\lambda > 0$.

$\lambda > 0$ In general $\mathcal{E}_\lambda$ is noncompact. However you can take L^2 extrinsic metric completion $\bar{\mathcal{E}}_\lambda$.

$\bar{\mathcal{E}}_\lambda$ is compact, $\partial \mathcal{E}_\lambda = \bar{\mathcal{E}}_\lambda - \mathcal{E}$ are orbifold singular E metrics. (V, g_∞).

g_∞ is C^∞ smooth away from a finite set of points $Q = \bigcup_i q_i$.

$M(q_i) \simeq C(S^3/\Gamma)$ $\qquad \Gamma \subset SO(4)$.

Q: Is $\bar{\mathcal{E}}_\lambda$ real analytic variety a "cycle"?

$\lambda = 0$ $\mathcal{E}_0$ — $\lambda=0$ component of $\mathcal{E}$

L^2 completion $\overline{\mathcal{E}_0} = \mathcal{E}_0 \cup \mathcal{E}_0^s$

complete, noncompact.

At ∞ in $\overline{\mathcal{E}_0}$ the E-metrics collapse in Cheeger Gromov sense outside a finite # of singular pts.

$\forall x \in M$, $\mathrm{inj}_{g_i}(x) \longrightarrow 0$, $\mathrm{inj}^2_{y_i}(y)|Rm|_{g_i}(y) \longrightarrow 0$ $\forall y$ away from finite number of pts

This was improved by Cheeger & Tian in '06, $|Rm|_{g_i}(y) \leq K$.

MAIN EXAMPLE: moduli space of K3 surfaces. (large cx structural limit)

$\lambda < 0$ $\mathcal{E}_\lambda$, $\overline{\mathcal{E}_\lambda} =$
- $\mathcal{E}_\lambda$
- $\mathcal{E}_\lambda^{orb}$ - orbifold
- cusps

$\uparrow$ L^2 completion

Cheeger & Tian proved no collapse.

Question: Is $\overline{\mathcal{E}_\lambda}$ compact?

Next time. some aspects for methods of proof, how orbifold singularities arise existence statements.