POSITIVELY & NON-NEGATIVELY CURVED MANIFOLDS WITH SYMMETRY

cmplt mfld $\Rightarrow$ universal cover is $\mathbb{R}^n$

sec ≤ 0 : Hadamard Cartan $\leftarrow$ mflds completely classified

sec>0, sec≥ 0 these prove to be mysterious

sec>0 : dim ≥ 24, $\pi_1 = 0$, S^n, $\mathbb{C}P^m$, $\mathbb{H}P^*$

sec≥ 0 : many, many constructions, much more known examples
e.g. products of positively curved mflds

Today don't have many known topological obstructions. Introduce 6 main thms.

① Gromov's Betti # bound.
M^n cmpt, sec≥ 0, $\Rightarrow \exists C(n)$ s.t. $\sum_i b_i(M^n, \#) \leq C(n)$

② Cheeger Gromoll Splitting Theorem
M^n cmpt, sec≥ 0, $\exists G \leq \pi_1(M)$, G abelian $\ni [\pi_1(M) : G] > \infty$
$\pi_1(M) / F_{\leftarrow \text{finite gp}} = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \times \Gamma$ $\leftarrow$ finite Abelian

③ Lichnerowicz, Hitchin,
M^n. cmpt spin, $\hat{A}^{\text{genus}} \neq 0$, $\alpha^{\text{invariant}} \neq 0$. $\Rightarrow$ sec ≥ 0 metric choices do NOT exist.

④ Cheeger Gromoll "Soul Thm"
M^n noncmpt, sec≥ 0, complete : $\exists$ totally geodesic submfld S & M is diffeo to the normal bundle over S.

sec>0

⑤ Bonnet-Myers
M^n, sec $\geq k > 0 \Rightarrow M$ cpt & $\pi_1(M) < \infty$

⑥ Synge : if $n = 2m \Rightarrow \pi_1(M) = \begin{cases} 0 & \text{orientable} \\ \mathbb{Z}_2 & \text{O.N.} \end{cases}$
M^n, sec $\geq k > 0$
if $n = 2m+1 \Rightarrow M^n$ is orientable

<u>Hopf Conjecture</u> $S^2 \times S^2$ does not admit g s.t. sec>0.

$S^2 \times S^2$ $\leftarrow$ if pos. curved
$\downarrow \pi$ then π is $\quad \pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$, use Synge thm
$\mathbb{R}P^2 \times \mathbb{R}P^2$ Not a submersion

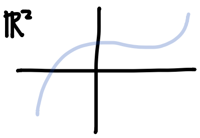
Thm (Hsiang, Kleiner '89)
Let T^1 act isometrically & effectively on M^4, $\pi_1(M^4) = 0$, sec$M^4 \geq k > 0$.
$\Rightarrow M^4$ is homeomorphic to S^4 or $\mathbb{C}P^2$
Recently improved to equivariant diffeomorphism by Grove-Wilking (another Spindeler)

Thm (S., Yang)
Let $T^1 \curvearrowright M^4$, $\pi_1(M^4) = 0$, M^4 closed & sec$\geq 0 \Rightarrow M^4$ is homeo to S^4, $\mathbb{C}P^2$, $S^2 \times S^2$ or <u>$\mathbb{C}P^2 \# \pm \mathbb{C}P^2$</u>
Recently improved to eq. diffeo by Grove-Wilking.

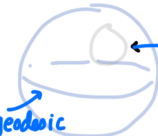
Lemma (Synge) improves on Berger Weinstein
Let $T^k \curvearrowright M^n$, sec $M^n \geq k > 0$.
(i) if $n = 2m$ then T^k has a fixed point
(ii) If $n = 2m+1$ then T^k has a circle orbit

Properties of Fixed pt sets
$N \subseteq Fix(M^n, S^1)$, M^n orientable

Then N is an orientable submfld of even codimension & <u>totally geodesic</u>



totally geodesic is rare in nature for $\mathbb{R}^2$



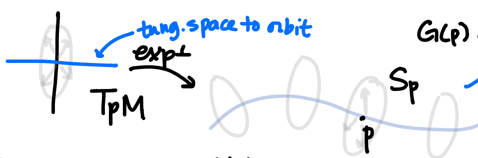
Totally geod. submflds share topolical properties w/ambient mfld

Proof of HK result : $\chi(M) = \chi(Fix(M, S^1))$, $2 \le \chi(M^4) \le 3$.

dim 4 $\Rightarrow$ even codim $\Rightarrow$ $\begin{cases} \text{Case 1: } \dim(Fix(M,S)) = 0. \\ \text{Case 2: } \dim(Fix(M,S^1)) = 2 \end{cases}$

Techniques Slice thm $G(p)$ orbit, G_p isotropy

Slice Theorem
technique of cmpt tfm gps



$G(p)$ orbit, G_p isotropy.

slice at p

use this to analyz quotient space.

orbit, exclude anything tangential.

Extent lemma $G \curvearrowright M$ & $M \to M/G$

For any choice of $(q+1)$ distinct pts $\bar{p}_0, \bar{p}_q \in X = M/G$ one has

$$\frac{1}{q+1} \sum_{i=0}^{q} xt_q S_{\bar{p}} X \underset{(\le)}{\ge} \pi/3 \text{ when curv } X \underset{(\le)}{\ge} 0.$$

extent (defined below) — unit space of directions at a pt — extent

$S_{\bar{p}} = S_p^{\perp}/G_p$, $\quad xt_q X = \max_{x_i, 1 \le i \le q} xt_q(x_1, \dots, x_q)$. ← maxml avg dist of q-tuples in X.

isotropy gp.

$xt_q : X^q \to \mathbb{R}$

$(x_1, \dots, x_q) \mapsto \binom{q}{2}^{-1} \sum dist(x_i, x_j)$.

Let's consider examples of extents.

Ex $xt_2 X = \operatorname{diam} X$

$$xt_q(S^1(1)) = xt_q(S^1(1)) = \begin{cases} \frac{p}{2p-1}\pi & q = 2p \\ \frac{p+1}{2p+1}\pi & q = 2p+1 \end{cases}$$

can also "pile up points" evenly to create other extents — on antipodes.

Case 1: Need to show there are at most 3 isolated fixed pts

$S^1 \times S^3(1) \to S^3(1)$ $\qquad S^3 \to X_{k,l}$

$(e^{i\theta}, z_1, z_2) \mapsto (e^{i\theta k} z_1, e^{i\theta l} z_2)$

$X_{1,1} = S^2(1/2)$, $S^2(1/2) \to X_{k,l}$

Assume $\exists$ 4 pts, $1/4 \sum xt_3 X_{k,l} \le \frac{1}{4} \sum xt_3 S^2(\frac{1}{2}) = \frac{1}{4} 4 \pi/3$ $\times$

Case 2 $\dim(Fix(M^4, S^1)) = 2$. WTS Either $N^2 = S^2$ or $S^2 \cup \{p\}$.

Soul Thm (Perelman)

Let X^n be a cmpt, finite-diml Alexandrov space with curv ≥ 0. Then there exists a convex cpt subset $S \subseteq X^n$, $\partial S = \emptyset$ and S a deformation retract of X^n. If curv $\ge k > 0$ then $S = \{p\}$

use thm to apply in case of quotient spaces

Soul Lemma (Grove, S-)

Suppose M is a closed mfld with $\sec M > 0$ & $G \curvearrowright M$ (G cpt Lie gp) $\ni \partial M/G \ne \emptyset$

Then

i) $\exists!$ $G(p) \subseteq M$ at maximal distance from $M_0 \subseteq M$.

ii) $M \cong D(M_0) \cup_E D(G(p))$

Maximal Symmetry Rank Thm (Grove, S-)

Let $T^k \curvearrowright M^n$, M^n closed, $\sec > 0$. Then

(i) $k \le [^{n+1}/_2]$, (ii) in the case of equality M^n is diffeo to S^n, $\mathbb{RP}^n$, $L_{p,q}$ or $\mathbb{CP}^m$.

Fixed pt homogeneous mflds

cohom fix $(M,G) = \dim(M/G) - \dim M^G - 1 \geq 0$. (=0 ⇒ Fixed pt homogeneous, FPH).
FPH, sec > 0, $\pi_1 = 0$ (Grove, S-)
⇒ CROSS S^n, $\mathbb{H}P^m$, $\mathbb{C}P^k$, $\mathrm{Ca}P^2$.