Metric measure spaces with Ricci lower bounds, Lecture 1

Andrea Mondino (Zurich University)

MSRI-Berkeley

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Motivation-Smooth setting:Comparison geometry

Question: (M, g) smooth Riemannian N-manifold.
If we assume some upper/lower bounds on the sectional or on the Ricci curvature what can we say on the analysis/geometry of (M, g)?

- Upper/Lower bounds on the sectional curvature are strong assumptions with strong implications E.g. Cartan-Hadamard Theorem (if $K \leq 0$ then the universal cover of M is diffeomorphic to $\mathbb{R}^N$), Topogonov triangle comparison theorem($\rightsquigarrow$ definition of Alexandrov spaces: non smooth spaces with upper/lower bounds on the "sectional curvature"), etc.
- Upper bounds on the Ricci curvature are very (too) weak assumption for geometric conclusions. E.g. Lokhamp Theorem (Gao-Yau, Brooks in dim 3): any compact manifold carries a metric with negative Ricci curvature.

Motivation-Smooth setting:Comparison geometry

Lower bounds on the Ricci curvature: natural framework for comparison geometry. E.g. Bishop-Gromov volume comparison, Laplacian Comparison, Cheeger-Gromoll splitting, Li-Yau inequalities on heat flow, etc.

A fundamental tool in the smooth setting is the Bochner identity: if $f \in C^\infty(M)$ then

$$\frac{1}{2}\Delta|\nabla f|^2 = |Hess\, f|^2 + Ric(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f).$$

If $dim(M) \leq N$ and $Ric \geq K\, g$ then Dimensional Bochner inequality, also called dimensional Bakry-Emery condition BE(K,N)

$$\frac{1}{2}\Delta|\nabla f|^2 \geq \frac{1}{N}|\Delta f|^2 + K|\nabla f|^2 + g(\nabla \Delta f, \nabla f).$$

Non smooth setting: Origins of the topic

Gromov in the '80ies

- introduced a notion of convergence for Riemannian manifolds, known as Gromov-Hausdorff convergence (for non-compact manifolds, more convenient a pointed version, called pointed Gromov-Hausdorff convergence $\rightsquigarrow$ GH-convergence of metric balls of every fixed radius)
- observed that a sequence of Riemannian N-dimensional manifolds satisfying a uniform Ricci curvature lower bound is precompact, i.e. it converges up to subsequences to a possibly non-smooth limit space (called, from now on, Ricci limit space)

• Natural question: what can we say about the compactification of the space of Riemannian manifolds with Ricci curvature bounded below (by, say, minus one)?
•Hope: useful also to establish properties for smooth manifolds.

Semi-smooth setting

- Cheeger-Colding 1997-2000 three fundamental papers on JDG on the structure of Ricci limit spaces.
  - Collapsing: $\liminf_k vol_{g_k}(B_1(\bar{x}_k)) = 0 \rightsquigarrow$ loss of dimension in the limit. More difficult, nevertheless they proved that the limit space has a uniquely defined volume measure (up to scaling) and a.e. point has a euclidean tangent space (the dimension may vary from point to point). Such points are called regular points, the complementary is called singular set.
  - Non collapsing: $\liminf_k vol_{g_k}(B_1(\bar{x}_k)) > 0$. More results: the Hausdorff dimension passes to the limit one can prove finer estimates on the singular set, e.g. Haudorff codimension 2.
- Colding-Naber, Annals of Math. 2012: the dimension of the tangent space does not change on the regular set, even in the collapsed case.
- Cheeger-Naber '14, proof of the codimension 4 conjecture (Anderson-Cheeger-Tian 1989): if (M_k^N, g_k) has uniform two-sided bounds on the Ricci curvature and is non-collapsing, then the singular set in the limit space has Haus-codim 4.

Extrinsic Vs Intrinsic

- The approach of Gromov-Cheeger-Colding to Ricci curvature for non-smooth spaces is an extrinsic point of view: consider the non smooth spaces arising as limits of smooth objects. Dichotomy collapsing-non collapsing. Very powerful for local structural properties.
- Analogy: like defining $W^{1,2}$ as completion of C^∞ endowed with $W^{1,2}$-norm.
- But $W^{1,2}$ can be defined also in completely intrinsic way without passing via approximations (very convenient for doing calculus of variations).
- GOAL: define in an intrisic-axiomatic way a non smooth space with Ricci curvature bounded below by K and dimension bounded above by N (containing ricci limits no matter if collapsed or not).
  $\rightsquigarrow$ analogy with GMT (currents, varifolds,etc.)

Preliminary Observation

- sectional curvature bounds for non smooth spaces make perfect sense in metric spaces (X, d) (Alexandrov spaces): sectional curvature is a property of lengths (comparison triangles)
- Ricci curvature is a property of lenghts and volumes: needs also a reference volume measure
  $\rightsquigarrow$ natural setting metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$.

Non smooth setting 1: the Kantorovich-Wasserstein space

Notations:

- $(X, \mathsf{d}, \mathfrak{m})$ complete separable metric space with a σ-finite non-negative Borel measure $\mathfrak{m}$ (more precisely $\mathfrak{m}(B_r(x)) \le c e^{Ar^2}$); if we fix a point $\bar{x} \in X$, $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ denotes the corresponding pointed space.
- Let
$$\mathcal{P}_2(X) := \{\mu \,:\, \mu \ge 0,\ \mu(X) = 1,\ \int_X \mathsf{d}^2(x, \bar{x})\, \mu(dx) < \infty\}$$
=Probability measures with finite second moment.
- Given $\mu_1, \mu_2 \in \mathcal{P}_2(X)$, define the (Kantorovich-Wasserstein) quadratic transportation distance
$$W_2^2(\mu_1, \mu_2) := \inf \left\{ \int_{X \times X} \mathsf{d}^2(x, y)\, \gamma(dxdy) \right\}$$
where $\gamma \in \mathcal{P}(X \times X)$ with $(\pi_i)_\sharp \gamma = \mu_i, i = 1, 2$
- $(\mathcal{P}_2(X), W_2)$ is a metric space, geodesic if (X, d) is geodesic

Non smooth setting 2: Entropy functionals

- On the metric space $(\mathcal{P}_2(X), W_2)$ consider the Entropy functionals $\mathcal{U}_{N,\mathfrak{m}}(\mu)$ if $\mu << \mathfrak{m}$

$$\mathcal{U}_{N,\mathfrak{m}}(\rho\mathfrak{m}) := -N\int \rho^{1-\frac{1}{N}}\,d\mathfrak{m} \quad \text{if } 1 \leq N < \infty \quad \text{Reny Entropy}$$

$$\mathcal{U}_{\infty,\mathfrak{m}}(\rho\mathfrak{m}) := \int \rho\log\rho\,d\mathfrak{m} \quad \text{Shannon Entropy}$$

(if μ is not a.c. then if $N < \infty$ the non a.c. part does not contribute, if $N = +\infty$ then set $\mathcal{U}_{\infty,\mathfrak{m}}(\mu) = \infty$.)

Non smooth setting: intrinsic-axiomatic definition. 2

- Crucial observation (Sturm-Von Renesse '05) If $(X, \mathsf{d}, \mathfrak{m})$ is a smooth Riemannian manifold (M, g), then $Ric_g \geq 0$ $(\geq Kg)$ iff the entropy functional $\mathcal{U}_{\infty,\mathfrak{m}}$ is (K-)convex along geodesics in $(\mathcal{P}_2(X), W_2)$. i.e. for every $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ such that for every $t \in [0,1]$ it holds

$$\mathcal{U}_{\infty,\mathfrak{m}}(\mu_t) \leq (1-t)\mathcal{U}_{\infty,\mathfrak{m}}(\mu_0)+t\mathcal{U}_{\infty,\mathfrak{m}}(\mu_1)-\frac{K}{2}t(1-t)W_2(\mu_0,\mu_1)^2.$$

- But the notion of (K-)convexity of the Entropy is purely of metric-measure nature, i.e. it makes sense in a general metric measure space $(X, \mathsf{d}, \mathfrak{m})$.
- DEF of $CD(K, N)$ condition [Lott-Sturm-Villani '06]: fixed $N \in [1, +\infty]$ and $K \in \mathbb{R}$, $(X, \mathsf{d}, \mathfrak{m})$ is a $CD(K, N)$-space if the Entropy $\mathcal{U}_{N,\mathfrak{m}}$ is K-convex along geodesics in $(\mathcal{P}_2(X), W_2)$ (for finite N is a "distorted" K-geod. conv.).

Non smooth setting: intrinsic-axiomatic definition. 3

Keep in mind:
- $CD(K, N) \rightsquigarrow$ definition Ricci curvature $\geq K$ and dimension $\leq N$ in an intrinsic/axiomatic way for metric measure spaces
- the more convex is $\mathcal{U}_{N,\mathfrak{m}}$ along geodesics in $(\mathcal{P}_2(X), W_2)$, the more the space is positively Ricci curved.

Good properties:

- CONSISTENT: (M, g) satisfies $CD(K, N)$ iff $Ric \geq K$ and $dim(M) \leq N$
- GEOMETRIC PROPERTIES: Brunn-Minkoswki inequality, Bishop-Gromov volume growth, Bonnet-Myers diameter bound, Lichnerowictz spectral gap, etc.
- STABLE under convergence of metric measure spaces? RK: stability will imply that Ricci limits are CD spaces.

Stability of $CD(K,N)$

- Lott-Villani '06: $CD(K,N)$ is stable under pmGH-convergence if the spaces are proper (i.e. bounded closed sets are compact). $CD(K,N)$, for $N<\infty$, implies properness but $CD(K,\infty)$ does not imply any sort of compactness not even local of the space. $\rightsquigarrow$ the result is satisfactory for $N<\infty$ (Ricci limits of bounded dimension are $CD(K,N)$) but not so much for $N=\infty \rightsquigarrow$ it does not cover sequences of Riemannian manifolds with diverging dimensions
- Sturm '06: $CD(K,N)$ is stable under $\mathbb{D}$-convergence if the reference measures are probabilities.
  $\rightsquigarrow$ does not cover blow ups (i.e. tangent cones) or blow downs (tangent cones at ∞)

Q:1) What is a natural notion of convergence if we drop properness of (X,d) and finitess of $\mathfrak{m}(X)$?
2) Is $CD(K,\infty)$ stable w.r.t. this notion?

Pointed measured Gromov (pmG for short) convergence

DEF:(Gigli-M.-Savaré '13) $(X_n, \mathrm{d}_n, \mathfrak{m}_n, \bar{x}_n) \to (X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in pmG-sense if there exist a complete and separable space (Z, d_z) and isometric embeddings $\iota_n : X_n \to Z$, $n \in \bar{N} := \mathbb{N} \cup \{\infty\}$ s.t.

$$\int \varphi\,(\iota_n)_\sharp \mathfrak{m}_n \to \int \varphi\,(\iota_\infty)_\sharp \mathfrak{m}_\infty, \ \forall \varphi \in C_{bs}(Z), \ \text{where}$$

$C_{bs}(Z) := \{f : Z \to \mathbb{R}$ cont., bounded with bounded support $\}$.

- The definition above is extrinsic but we prove it can be characterized in a (maybe less immediate) totally intrinsic way, according various equivalent approaches (via a pointed version of Gromov reconstruction Theorem or via a pointed/weighted version of Sturm's $\mathbb{D}$-distance).
- On doubling spaces pmG-convergence above is equivalent to mGH-convergence ($\rightsquigarrow$ consistent with Lott-Villani).
- On normalized spaces of finite variance pmG-convergence is equivalent to $\mathbb{D}$-convergence ($\rightsquigarrow$ consistent with Sturm).
- pmG-convergence no a priori assumption on $(X_n, \mathrm{d}_n, \mathfrak{m}_n)$.

$CD(K,\infty)$ is stable under pmG-convergence

THM(Gigli-M.-Savaré '13): Let $(X_n, \mathrm{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N}$, be a sequence of $CD(K,\infty)$ p.m.m. spaces converging to $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmG-sense. Then $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ is a $CD(K,\infty)$ space as well.

Idea of Proof:

1. prove Γ-convergence of the entropies under pmG-convergence
2. use the compactness of $\mathfrak{m}_n$ to prove compactness of Wasserstein-geodesics in the converging spaces
3. conclude that K-geodesic convexity is preserved.

□

Non completely satisfactory feature of $CD(K, N)$

- The stability implies that the class of $CD(K, N)$ spaces is closed under pmG-convergence and in particular contains Ricci limits. But is it the smallest one doing such a job?
- The class of $CD(K, N)$ spaces is TOO LARGE: compact Finsler manifolds satisfy $CD(K, N)$ for some $K \in \mathbb{R}$ and $N \geq 1$ [Ohta] (earlier work in this direction by Cordero-Erasquin, Sturm and Villani), but if a smooth Finsler manifold M is a mGH-limit of Riemannian manifolds with $Ric \geq K$ then M is Riemannian (Cheeger-Colding '00).
- $\rightsquigarrow$ We would like to reinforce the $CD(K, N)$ condition in order to rule out Finsler structures, but in a sufficiently weak way in order to still get a STABLE notion (so to include Ricci limit spaces).

Cheeger energy and Heat flow in m.m.s

Given a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ and $f \in L^2(X, \mathfrak{m})$, define the Cheeger energy

$$Ch_{\mathfrak{m}}(f) := \frac{1}{2}\int_X |\nabla f|_w^2 \, d\mathfrak{m} = \liminf_{u \to f \; in L^2} \frac{1}{2}\int_X (\mathrm{lip} u)^2 d\mathfrak{m}$$

where $|\nabla f|_w$ is the minimal weak upper gradient.
Two ways to define the heat flow

- either as gradient flow in $L^2(X, \mathfrak{m})$ of $Ch_{\mathfrak{m}}$
- or as gradient flow in $(\mathcal{P}_2(X), W_2)$ of $\mathcal{U}_{\infty,\mathfrak{m}}$

THM(Ambrosio-Gigli-Savaré 2011) For arbitrary m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ satisfying $CD(K, \infty)$ the two approaches coincide.

RK: in $\mathbb{R}^n$ proved by Jordan-Kinderlehrer-Otto, in Riemannian manifolds (M, g) proved by Ohta, Savaré, Villani, Erbar, in Alexandrov spaces by Gigli-Kuwada-Ohta

The $RCD(K, N)$ condition

Crucial observation: On a Finsler manifold M, the Cheeger energy is quadratic (i.e. parallelogram identity holds) iff the heat flow is linear iff M is Riemannian.

Definition [Ambrosio-Gigli-Savaré 2011, Ambrosio-Gigli-M.-Rajala 2012, Erbar-Kuwada-Sturm 2013, Ambrosio-M.-Savaré 2015]
Given $K \in \mathbb{R}$ and $N \in [1, \infty]$, $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD(K, N)$ space if it is a $CD(K, N)$ space & the Cheeger energy is quadratic (or, equivalently, CD(K,N) & linear heat flow).

Q: is RCD stable under convergence of m.m.s. (crucial in order to say that Ricci limits are RCD)?

Stability of $RCD(K, N)$

THM (Ambrosio-Gigli-Savaré '11 and Gigli-M-Savaré '13): Let $(X_n, \mathrm{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N}$, be a sequence of $RCD(K, N)$ p.m.m. spaces, $K \in \mathbb{R}$, $N \in [1, \infty]$, converging to a limit space $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmG-sense. Then $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ is $RCD(K, N)$ as well.

Idea of proof:
Step 1) we already know that $CD(K, N)$ is stable, so $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ is a $CD(K, N)$ space.
Step 2) Heat flow on $(X_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty)$ is linear
i) Heat flows are stable under pmG conv.+CD (via identification of H_t with gradient flow of the entropy in $(\mathcal{P}_2, W_2)$)
ii) Since the heat flows of X_n are linear, by the stability of heat flows also the limit heat flow is linear. $\square$

Examples of RCD-spaces

- Ricci limits, no matter if collapsed or not and no matter if the dimension is bounded above or not (in the first case get $RCD(K,N)$, in the latter get $RCD(K,\infty)$)
- Finite dimensional Alexandrov spaces with curvature bounded below (Perelman 90'ies and Otsu-Shioya '94: Ch is quadratic, Petrunin '12: CD is satisfied)
- Weighted Riemannian manifolds with $N - Ricci \geq K$: i.e. (M^n, g) Riemannian manifold, let $\mathfrak{m} := \Psi\, vol_g$ for some smooth function $\Psi \geq 0$, then $Ric_{g,\Psi,N} := Ric_g - (N-n)\frac{\nabla^2 \Psi^{1/N-n}}{\Psi^{N-n}} \geq Kg$ iff $(M, \mathrm{d}_g, \mathfrak{m})$ is $RCD(K,N)$.
- Cones or spherical suspensions over RCD spaces (Ketterer '13)
- Wiener space (Ambrosio-Erbar-Savaré '15)

Bochner inequality

- We say that $(X, \mathrm{d}, \mathfrak{m})$ has the Sobolev-to-Lipschitz (StL) property if
  $\forall f \in W^{1,2}(X)$, $|\nabla f|_w^2 \leq 1 \Rightarrow f$ has a 1-Lipschitz repres.
- $RCD(K, \infty)$ implies the StL-property (Ambrosio-Gigli-Savaré).
- We say that $(X, \mathrm{d}, \mathfrak{m})$ satisfies the dimensional Bochner Inequality, $BI(K, N)$ for short, if
  -$Ch_{\mathfrak{m}}$ is quadratic & StL-property holds,
  -$\forall f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ with $\triangle f \in L^2(X, \mathfrak{m})$ and $\forall \psi \in LIP(X)$ with $\triangle \psi \in L^\infty(X, \mathfrak{m})$ it holds

$$\int_X \left[\frac{1}{2}|\nabla f|_w^2 \Delta \psi + \Delta f \, div(\psi \nabla f)\right] d\mathfrak{m} \geq K \int_X |\nabla f|_w^2 \psi d\mathfrak{m} + \frac{1}{N}\int_X |\Delta f|^2 \psi d\mathfrak{m}.$$

$RCD(K,N)$ is equivalent to $BI(K,N)$

THM(Erbar-Kuwada-Sturm and Ambrosio-M.-Savaré)
$(X, \mathrm{d}, \mathfrak{m})$ satisfies the dimensional Bochner inequality $BI(K,N)$ iff it is an $RCD(K,N)$ space.

- the approach of EKS is based on the equivalence of an entropic curvature condition involving the Boltzman entropy and uses a weighted heat flow (which is linear)
- the (subsequent and independent) proof by AMS involves non linear diffusion equations in metric spaces: more precisely the porous media equation (which is the nonlinear gradient flow of the Renyi entropy) plays a crucial role in the arguments.
- the case $N = \infty$ was already established by Ambrosio-Gigli-Savaré '12 (based also on previous work by Gigli-Kuwada-Ohta 2010)

Some analytic properties of $RCD(K, N)$ spaces

- Local Poincaré inequality (Cheeger-Colding '00, Lott-Villani '07, Rajala '12)
  On $CD(0, N)$:
  $\fint_{B_r(x)} |u - \langle u \rangle_{B_r(x)}| \, d\mathfrak{m} \leq 2^{N+2} r \fint_{B_{2r}(x)} |\nabla u|_w \, d\mathfrak{m}$
- Li-Yau and Harnack type inequalities (Garofalo-M. '13):
  On $RCD(0, N)$: $\Delta(\log(H_t f)) \geq -\frac{N}{2t} \quad \mathfrak{m}\text{-a.e.} \quad \forall t > 0$
  On $RCD(K, N)$, $K > 0$: $\forall x, y \in X$ and $0 < s < t$
  $$(H_t f)(y) \geq (H_s f)(x)\, e^{-\frac{\mathsf{d}^2(x,y)}{4(t-s)e^{\frac{2Ks}{3}}}} \left(\frac{1 - e^{\frac{2K}{3}s}}{1 - e^{\frac{2K}{3}t}} \right)^{\frac{N}{2}}.$$
- Laplacian comparison (Gigli '12):
  On $RCD(0, N)$: $\Delta \mathsf{d}(x_0, \cdot) \leq \frac{N-1}{\mathsf{d}(x_0, \cdot)}$
- Harmonic functions with polynomial growth
  (Colding-Minicozzi '97 and Hua-Kell-Xia '13)
  On $RCD(0, N)$: $\dim H^k \leq C k^{N-1}$,
  where
  $H^k = \{u : X \to \mathbb{R} \text{ s.t. } \Delta u = 0, \, |u(x)| \leq C_0(1 + \mathsf{d}(x, x_0))^k\}$.

Some geometric properties of $RCD(K, N)$ spaces

- Splitting Theorem (Cheeger-Gromoll '71, Cheeger-Colding '96, Gigli '13)
  If $(X, \mathsf{d}, \mathfrak{m})$ is $RCD(0, N)$ and X contains a line then
  $X \simeq Y \times \mathbb{R}$ as m.m.s.
- Euclidean Tangents (Cheeger-Colding '97, Gigli-M.-Rajala '13, M.-Naber '14)
  If $(X, \mathsf{d}, \mathfrak{m})$ is $RCD(K, N)$ then $\mathfrak{m}$-a.e. $x \in X$ has a unique euclidean tangent cone of dimension $n(x) \leq N$.
  Moreover called $A_k = \{x \in X : T_x = \mathbb{R}^k\}$ we have that A_k is k-rectifiable$\rightsquigarrow$ stratification into rectifiable strata
- Maximal diameter (Cheng '75, Ketterer '14)
  If $(X, \mathsf{d}, \mathfrak{m})$ is $RCD(N-1, N)$ and $\mathrm{diam} X = \pi$ then
  $X \simeq [0, \pi] \times_{sin}^{N-1} Y$ as m.m.s.

What to remember from this lecture

- If (M_i, g_i) is a sequence of Riemannian manifolds with $\dim M_i \le N$ and $Ric_{g_i} \ge K g_i$ converging in mGH sense to a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ then $(X, \mathrm{d}, \mathfrak{m})$ is an $RCD(K, N)$ space, no matter if collapsing or not. If the dimensions $\dim M_i \to \infty$ the limit space is $RCD(K, \infty)$.
- Most of the results in Riemannian geometry classically known for manifolds with $Ric_g \ge Kg$ are now settled also for $RCD(K, N)$ spaces.
- Tomorrow: Levy-Gromov isoperimetric inequality in $RCD(K, N)$ setting.

!!THANK YOU FOR THE ATTENTION!!