A Panoramic Glimpse of Non-Negative Curvature

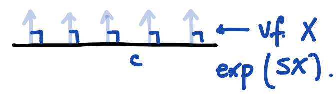
We'll go back in time & explore tools & structures of the past

M^n : complete Riemannian manifold, $\sec M \geq 0$
$\langle , \rangle$: inner product on M

Early link between geometry & topology: <u>Morse Theory</u>

Let's give another way to determine if $\sec M \geq 0$.
Invoke parallel transport:

$\sec M \geq 0 \iff \forall\, c : [0,1] \to M$ (a geodesic), $\forall X \perp c$, parallel along c,
The index form $I(x,x) := \int_0^1 \langle x', x' \rangle - \langle R(x,\dot{c})\dot{c}, X \rangle \leq 0$.



Note $I(x,x) = \frac{d}{ds} L(c_s)\big|_{s=0}$
↖ length functional on family of geodesics

Baby Morse Theory of Geodesics

$$\begin{array}{ccc} \Omega & & \\ M^I_N & \subseteq & M^I \ni c \\ \downarrow & & \downarrow \\ N & \subseteq & M \times M \ni (c(0), c(1)). \end{array}$$

Take submanifold $N \subseteq M \times M$, update (in blue)

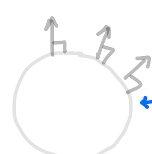
Bonnet Myers Thm

$\sec M \geq 1$
$\Downarrow$
$\operatorname{diam} M \leq \pi$
$N = \{p\} \times \{q\}$.

Thm (Synge) $N = \Delta(M)$ ← take space of closed curves
↖ diagonal

Assume $\sec M^{2n} \geq 1$, M orientable
$\Downarrow$
Then $\pi_1(M) = \{1\}$

get some parallel vector field $\Rightarrow$ fixed pt.

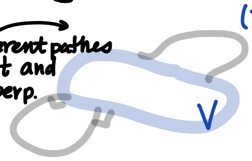


Thm (Frankel) Assume $N = V^a \times W^b$ $a + b \geq n$
(NOT ALL DETAILS)
totally geodesic submanifolds

<u>Then</u> $V \cap W \neq \emptyset$.

Idea: Index argument

Thm (Wilking) $N = V \times V$
(This is fairly recent result in terms of mathematics)



$$\begin{array}{ccccc} \text{have index} \to V & \subseteq & \overset{\text{path space}}{M^I_{V\times V}} & \xrightarrow[E]{\text{energy}} & \mathbb{R} \\ & & \downarrow & & \\ & & V \times V & & \end{array}$$

consists of all length zero curves

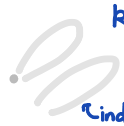
<u>Sketch of Pf</u> | Look at 1st nontrivial geodesic, use morse theory
Topology changes, since index changes. add a cell
For homotopy gps below the dimension of the cell you add, they look the same

Berger

$\sec M \geq 1$, $\operatorname{diam} M > \pi/2$

$M^n \sim S^n$

Argument again by Morse theory.
Take geodesic, want to compute index
know the new geodesic is "long"



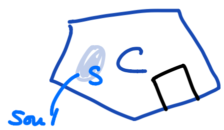
Ω ← loop space of mfld

index $m-1$

Conj. (Bott) ($\sec M \geq 0$) $\Rightarrow$ M is "topologically elliptic"

don't know if it's true → [$B(\Omega M)$] (the betti numbers of the loop space) grows at most polonomially (very strong)

Convexity is very important for nonneg. curvature in various situations

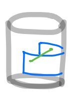


$C \subseteq M$ convex (loc convex)

$\sec M \geq 0, \partial C \neq \emptyset \Rightarrow \operatorname{dist}(\partial C, \cdot)$ is concave

If $\sec M > 0$, then the concavity is strict

this is basis for the Soul Thm.



It's not convex but it is Locally convex

Soul THM (Cheeger-Gromoll)

M, $\sec M \geq 0$, complete, noncompact

$\Downarrow$

$\exists S \subseteq M : T^\perp S \overset{\text{diffeo.}}{\sim} M$

cpt tot convex submanifold

Normally you don't expect to find compact sets

Consider if we have  dist. func is concave but we can look at subsets

Note: the sphere is an "extreme" manifold in many ways.

CONSTRUCTIONS & EXAMPLES.

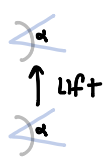
Main Source: G a compact Lie gp w/ bilinear metric, $\sec G \geq 0$.

$$\begin{array}{c} M \\ \pi\downarrow \\ N \end{array} \quad \text{Riemannian submersion}$$

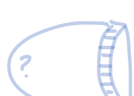
then $\sec M \geq 0$ $\Downarrow$ $\sec N \geq 0$

Pf idea

$$\begin{array}{c} M \\ \downarrow \\ N \end{array}$$



$\sec M \geq 0$, take products and apply special gluings



↙ may be able to paste boundaries together.

Cheeger used gluing to create many examples of projective spaces

G/H all have sec ≥ 0, G/H with sec > 0 classified

CROSS (compact rank one symmetric space)

homogenous spaces occur in dim 6, 7, 12, 13, 29

# of constructions 1, ∞, 1, 1, 1

$H \subseteq G \times G$, many have sec > 0, dim: 6, 7, 13

constructions 1, ∞, 1

Describe/Classify mflds M with sec $M \geq 0$ & large or special isom. gp action.

(M^4, S^1), S^4, $\mathbb{C}P^2$, $S^2 \times S$, $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$

sec > 0

These are those w/ sphere gp action.

Lots of problems to concoct!

consider examples

<u>Large</u> here's an example of what this could mean

dim M/G is small ("small cohomogeneity").

Theorem related to this:

Thm (Wilking) sec $M^n > 0$, cohom $< k$.

If $n \geq 18(k+1)^2$ (if dim. is sufficiently large compared to cohomogeneity)

then $M \sim$ CROSS. "<u>much</u> better than homotopy equivalence."

<u>cohom 1</u>

Thm (Verdiani, Wilking, Grove, Ziller)

sec $M > 0$, cohom 1

$\Downarrow$

$M \sim$ CROSS (up to equivariant diffeo)

<u>or</u>

dim M = 7, 13 ← classification

↑ exhaustive list, ∞ new candidates

$M^7 \simeq T, S^4 \# \Sigma$ ← exotic structure

one new example (independently done by Deancolt, Grove liedicani, Ziller) (sp?)

Ends with discussion of special in terms of polar manifolds. (have polar action equipped)

Requires entirely different techniques and may admit interesting examples.

8 min

MSRI Talk 2

3 first pages on 20 min

1_2

A panoramic glimpse of nonnegative curvature

Throughout $(M, \overset{\langle,\rangle}{g})$ complete Riem. man.

① Index comparison

$\sec M \geq 0 \iff \forall c : [0,1] \to M$ geod

$\forall X \perp c$, parallel : $I(X,X) \underset{(=)}{\leq} 0$

Invoking parallel transport

$\Downarrow$

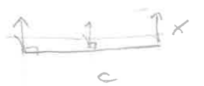
(dist. comparison)

2 min

$$\frac{1}{\ell(c)} \frac{d^2}{ds^2} L(c_s) = \int_0^1 \langle X', X' \rangle - \langle R(X, \dot c)\dot c, X \rangle$$

$$\langle R(X,\dot c)\dot c, X\rangle = \sec(X, \dot c)$$

$c_s = \exp(sX)$



$\sec > 0 \Rightarrow$

rigidity?

index $\geq$ dim parallel fields

3 min

② "Baby Morse theory of geodesics" up mid 70's

Geometry Topology

$$\begin{array}{ccccc} \mathbb{R}_{\geq 0} \xleftarrow{L,E} & M^I_N & \subset & M^I & c : [0,1] \to M \\ & \downarrow & & \downarrow & \downarrow \\ & N & \subset & M \times M & (c(0), c(1)) \end{array}$$

$\Omega M \to M^I$

fibration

Topology via Morse Theory

3 min

- $c \in M^I_N$ critical $\iff$ c geodesic & $(\dot c(0), -\dot c(1)) \perp N$

Classical applications :

Skip some?

- (BM) [1885–41] $\sec M^n \geq 1 \Rightarrow \operatorname{diam} M \leq \pi$ ($N = \{p\} \times \{q\}$) Schoenberg (32) ≡ Morse (34)
- Synge (36) — " — , n even $\Rightarrow \pi_1(M) = \begin{cases} 1 \\ \mathbb{Z}_2 \end{cases}$ $N = \Delta(M)$
- Frankel (61) — " — $V^a, W^b \subset M$ tot. geod, $a+b \geq n \Rightarrow V \cap W \neq \emptyset$ $N = V \times W$
- Wilking (2003) — " — $V^{n-k} \hookrightarrow M^n$ tot. geod $\boxed{n-2k+1 \text{ connected}}$ $N = V \times V$

higher index $\searrow$ Berger $\iff$ Bott

sin x

Gromoll (Past)

2_2

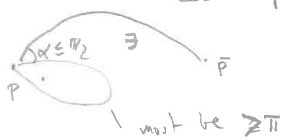
- Klingenberg - Cheeger Gromoll

  M 1-conn. $\frac{1}{4} \leq \sec \leq 1 \Rightarrow \operatorname{inj}(M) \geq \pi$ (by contradiction)

- Berger M 1-conn, $\sec M \geq 1$, $\operatorname{diam} > \pi/2$

  $\Rightarrow \Omega M = pt \cup e^{n-1} \cup \dots$

  index $\geq n-1$



(3 min)

[Bott Samelson G, Symmetric spaces]

Topology of Ω

Bott - Conjecture	M^n 1-conn, $\sec M \geq 0$ $\Downarrow$ M top. elliptic, i.e., $\{\beta_k(\Omega M)\}$ growth at most polyn.

—o—

Toponogov & Convexity (3)

$\rightsquigarrow$ Toponogov

(3 min)

C (loc.) convex

$\partial C \neq \emptyset$, $\sec M \geq 0$

$\Downarrow$ Cheeger - Gromoll

concave $\operatorname{dist}_{\partial C} : C \to \mathbb{R}$

"concave" (strict if $\sec > 0$)

[C-G] Soul thm

$\sec M \geq 0$, M complete, non-compact $\Downarrow$ $\exists\, S \subset M : M \underset{\text{diffeo}}{\sim} T^{\perp}S \to S$ cpt tot. convex sub man.

(h.e. by Morse Th.)

[Gromov] Betti #

$\exists\, C = C(n)$ s.t. $\dim H_*(M, F) \leq C$ any M^n $\sec M \geq 0$

5 min

3

$\exists C(n,k)$ diam rigidity scale M such that $\operatorname{diam} M = \pi$

κ 0 $1/4$ 1 min sec

sphere $S^n(1)$ rigid

[GG, W] $M \sim S^n$ or iso CROSS or S^n/Γ red. or $\mathbb{C}P^{odd}/\mathbb{Z}_2$.

[Convexity, crit pt theo, some topology]

nothing else known! — o —

4 Curvature pinching

$0 < \delta \leq \sec \leq 1$

0 $1/4$ 1

$\delta = 1/4 - \varepsilon$ o.k.

"many examples" but no general results

except

(Cheeger) (Klingenberg) $\operatorname{inj} \geq \pi$

• $\dim M$ even, δ-pinched $\Longrightarrow$ Finitely many diffeo

δ pinched FR

$\dim M$ arbitrary

$|\sec M| \leq 1$, $\operatorname{diam} \leq D$, M 2-connected $\pi_1 = \pi_2 = 0$

$\Downarrow$ Finitely many diffeo.

$\pi_2 \neq 0$ $\exists E_1, \dots, E_N$: $M = E_i/T^{k_i}$

Not

Klingenberg-Sakai ?

BUT NOT How to find new Examples!

Next 2 pages on 10 min

5 Construction & Examples

Main source: G Lie gp with biinvar metric

$\sec G \geq 0$

Main Construction:

projection $\begin{array}{c} M \\ \downarrow \pi \\ N \end{array}$ Riem subm. / in fact $\begin{array}{c} M \\ \downarrow \\ M/G \end{array}$ free G-action.

[exception $\begin{array}{c} S^{15} \\ \downarrow \\ \mathbb{O}P^2 \end{array}$]

in $\sec \geq 0$ **Products!**

Assoc. bundles $\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ B \end{array}$ $G \times F \to F$ $\begin{array}{c} P \times_G F \\ \downarrow \\ B \end{array}$

All G/H have $\sec \geq 0$

$\sec > 0$ classified: CROSS + dim: 6, 7, 12, 13, 24

1, ∞, 1, 1, 1

All $G/\!/H$ have $\sec \geq 0$

$\sec > 0$ many dims: 6, 7, 13

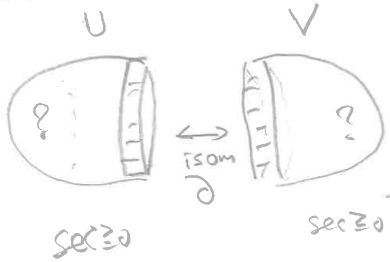
1, ∞, ∞

PROBLEM: Find M 1-conn. cpt. $\sec \geq 0$ with a metric with $\sec > 0$!?

- Hopf: $S^2 \times S^2$?
  - M Higher rank symmetric space
- No $M = V \times W$ is known to have $\sec > 0$

PROBLEM: $\exists N$ s.t. $M \sim$ CROSS if $\sec M > 0$ & 1-conn. $\dim M \geq N$?

Special Gluing Constructions



Note: both convex

Cheeger: CROSS # CROSS

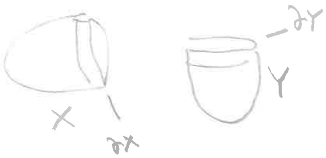
G-Ziller: M cohom 1

codim 2 sing. orbits

Example: $\partial(X \times Y) = \partial X \times Y \cup X \times \partial Y$

U V

$\partial U = \partial X \times \partial Y = \partial V$



Many more using manifolds with corners

Note $S^n(1)$ extreme rel to

- size
- pinching
- symmetry

→ diagonals

→ curv pinch

6_2

30 minutes for rest

6 Symmetry Program

$\boxed{\text{Describe/Classify manifolds } M \text{ with } \sec \geq 0 \text{ and large/special isom. group actions}}$

Now clear distinction between $\sec \geq 0$ & $\sec > 0$

even $\sec \gtrapprox 0$

$G \times M \to M$

Large: · $\dim G$ (rel. $\dim M$) — sym deg

· $\dim M/G$ small — cohom

$\overset{=}{[0, 1, \ldots, k, \ldots}$ ← Many, but Not all $\sec \geq 0$, all $\gtrapprox 0$!

all elliptic !

done ! — symrank : cofix

HK: $S^1 \times M^4 \to M^4$

THM (Wilking) $\sec M > 0$, r conn, cohom $(M, G) = k$. Then $n \geq 18(k+1)^2 \Rightarrow M \sim$ CROSS

$= 72$ if $k=1$ ... the (25 if $k=0$)

n even — n odd

THM (Verdiani, G-Wilking, Ziller)

$\sec M > 0$, 1-conn, cohom 1

$\Downarrow$

· $M \sim$ CROSS (equiv. diffeo) [Normal Homogeneous only two noncross hom.]

or

· $\dim M = 7, 13$ ← complete Classif. (Bazaikin)

∞ — ∞

Esch. ⊂ exhaustive list of CANDIDATES $P_1, P_2, P_3, \ldots$ Q_1, Q_2, Q_3

∞

one of them P_1 ✓

THM (Dearricott, G-Verdiani, Ziller)

P_1 has $\sec \geq 0$

S^1

$T_1 S^4 \# \Sigma^7$

NEW Example

New Construction !

Special actions

$\Updownarrow$

symmetric spaces

$G \times M \to M$ polar $\Leftrightarrow \exists$ section $\sigma : \Sigma \xrightarrow[\text{imm}]{} M$ $\pitchfork$ all orbits (tot. geod)

Examples : Extreme : G discrete / finite

(a) $\Sigma = M$

(b) G transitive : $\Sigma = pt$.

(c) G cohom 1, $\Sigma = \begin{cases} S^1 \\ \mathbb{R} \end{cases}$

- Ad : $G \times G \to G$ $\quad \Sigma =$ max torus. e.g. $G = U(n)$
- More general Isotopy actions / representations of symmetric spaces $G/K = M$

$K \curvearrowright T_{(K)}M$ $\circlearrowleft K$

THM (Fang, G, Thorbergsson)

M 1-conn, $\sec M > 0$, $G \times M \to M$ polar $\dim M/G \geq 2$

$\Downarrow$

$\underset{\substack{\text{equiv.}\\ \text{diffeo}}}{(M, G) \sim} (\text{CROSS}, G)$ ← polar action

Starting point M 1-conn.

$\Downarrow$ $G_\Sigma/\ker = W$ refl. gr. on Σ

(and $\Sigma/W = M/G$)

$\quad\quad\; \begin{matrix}\cong\\ C \subseteq \Sigma\end{matrix}$ chamber, convex

faces ~ mirrors

Note $M = G \cdot C$

$\mathcal{C}(M, C)$ chamber system.

THM $\sec \Sigma > 0$, W refl. gp.

$\Downarrow$ $(\Sigma^k, W) \sim$ (equiv. diff) $\begin{cases} (S^k, W) \text{ Coxeter} \\ \text{or induced on } (\mathbb{RP}^n, \ ) \end{cases}$

— o —

THM $\sec \Sigma \geq 0$, W cocompact refl. gp.

$\Downarrow$

$\tilde{\Sigma} = \mathbb{R}^e \times S^{k_1} \times \dots \times S^{k_r} \times \Theta \times N$ ← w/ triv action.

and

$\tilde{W} = W_0 \times \underbrace{W_1 \times \dots \times W_r}_{\text{finit Coxeter}} \times W_\Theta$

↑ affine Coxeter.

Θ product of open or iterated open books ⇝ unrelated to Symmetric Space

$S(\nu \oplus \varepsilon)^k =$ or, iterated $S(\nu_1 \oplus \varepsilon)^{k_1} \times \dots \times S(\nu_k \oplus \varepsilon)^{k_r}$ ↓ $S \times \dots \times S$

pullback ↓ S ↗

$D(\nu)$ ← soul.

∂D

$\Sigma = \partial D \times D^k \cup D(\nu) \times S^{k-1} = S(\nu \oplus \mathbb{R}^k)$

$\circlearrowleft W$ $\circlearrowleft W$

Example $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$

$\mathbb{HP}^2 \# \pm \mathbb{HP}^2$

$$\begin{array}{ccc} L \times G/K & \to & G/K \\ \uparrow & & \uparrow \\ (L \times K) \times G & \to & G \end{array}$$

$\boxed{\text{Irreducible}}$ i.e. $M/G = \Delta$ W irred Coxeter

$\Downarrow$ $\exists \ \hat{G} \curvearrowright P^L$ prin L bdl

$\downarrow$ P symmetric space

$G \curvearrowright M$

Full circle back to "Bott Samelson"!

7 Additional Tools

In M : • representation theory

$Gp = G/G_p$

$G \cdot p$

$p \cdot \circlearrowleft G_p$

$\mathcal{F}_G$ $\mathcal{F}_G^{\#}$ dual foliation

singular orbit "foliation"

$M \to M/G$

(Transverse Jac. field estimates.)

In M/G : Toponogov "even better"

Alex. — critical pt. theory.

Convexity, Souls etc now abundant

$\boxed{\text{when } \partial M/G \neq \emptyset \text{ " } \cup \text{ faces}}$

For polar actions

$\boxed{\text{Chamber systems \& Buildings: Bruhat-Tits}}$

w/ topology