COMPARISON GEOMETRY FOR RICCI CURVATURE II

(we will focus on extension to Bakry Emery Curvature)

Comparison to Bakry-Emery Ricci & integral Ricci
- Ricci for smooth metric space $(M^n, g, e^{-f} dvol_g)$
  this occurs as collapsed limit of smth mfld

$M^n \times F^m$ (warped product)

$g_\varepsilon = g_m + \varepsilon^2 (e^{-f/m})^2 g_F$

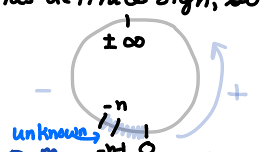
Consider $M^n \times F^m \xrightarrow[\text{renormalized measure GH limit}]{\widetilde{dvol_{g_\varepsilon}},\ \varepsilon \to 0} (M^n, e^{-f} dvol_g)$

In this warped product, what is Ricci curvature?

Ric of g_ε : $Ric_M + Hess f - \frac{1}{m} df \otimes df \triangleq Ric_f^m$ in M direction. this is "m-Bakry Ricci curvature"

we can take $m \in \mathbb{R}$. For $m=0$ assume f constant.

Note $df \otimes df$ has definite sign, so it will be monotone in m.



Ric_f^m is $\uparrow$ as m goes from $0^+ \to +\infty, -\infty \to 0^-$.

$Ric_f^m \geq Hg$ is weaker.

Note (update later) $Ric_f^m \geq 0 \longleftrightarrow RCD(0, n+m)$.

$Ric_f^m \geq 0 \leftrightarrow RCD(0, n+m)$. This is a special case for RCD

$Ric_f^m = \lambda g$ Quasi einstein equation

$m=0$ Einstein eqn

$m \in \mathbb{N}$ $M^n \times_{e^{-f/m}} F^m$ is Einstein (warped product Einstein).

$m = \infty$ Ric + Hess = λg gradient Ricci soliton

$m<0$ conformally Einstein.

Question: what topological & geometric results for Ric curvature can be extended to Ric_f^m

$0 < m < \infty$ $\qquad$ $m = \infty$ $\qquad$ $0 > m \geq -n+1$

For Ric, Bochner formula $\frac{1}{2}\Delta_f |\nabla u|^2 = |\underbrace{Hess u}_{\geq \frac{(\Delta u)^2}{n}}|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + Ric_f(\nabla u, \nabla u)$

$\Downarrow$

$= Ric_f^m + \frac{1}{m} \underbrace{df \otimes df}_{\langle \nabla f, \nabla u\rangle}$.

$(M, g, e^{-f} dvol_g)$ $\Delta_f u = \Delta u - \nabla f * \nabla u$ is self adjoint wrt $e^{-f} dvol_g$

we can use simple inequality

when $0 < m < \infty$, $\frac{a^2}{n} + \frac{b^2}{m} \geq \frac{(a+b)^2}{m+n}$ $\qquad$ also this still holds for $-\infty < m < -n$.

$$\frac{1}{2}\Delta_f |\nabla u|^2 \geq \frac{(\Delta_f u)^2}{n+m} + \langle \nabla u, \nabla \Delta_f u \rangle + \mathrm{Ric}_f^m(\nabla u, \nabla u)$$

Laplace comparism: $\mathrm{Ric}_f^m \geq (n+m-1)H$, then $\Delta_f r \leq \Delta_H^{n+m} r$ (Bakry, Qian 2005)

<u>Results extend directly</u> $m = \infty$.

$\boxed{m=\infty}$ Example: $\mathbb{H}^n$, $f(x) = (n-1)d^2(x,x)$.

$\mathrm{Ric}_f \geq n-1 > 0$.

Things don't extend directly

Thm (Wei-Wylie, 2009) If $\mathrm{Ric}_f \geq (n-1)H$

$$\Delta_f(r) \leq \Delta_H(r) - \frac{1}{sn_H^2(r)}\int_0^r (sn_H^2(t))' \partial_t f(t)\, dt \quad \text{with } sn_H'' + H sn_H = 0$$

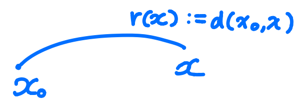
this is a distance function along a geodesic $\quad sn(0) = 0,\ sn'(0) = 1$

$(sn_H^2(\Delta r - \Delta_H r))' \leq sn_H^2\, \partial_r^2 f$

Cor: $|f| \leq k$. $\Delta_f(r) \leq \Delta_H^{n+4k}(r)$.

for example, when M is compact.

volume comparison.



When $n<0$ the inequality $\frac{a^2}{n} + \frac{b^2}{m} \geq \frac{(a+b)}{m+n}$ still holds

so the consequential inequality $\frac{1}{2}\Delta_f|\nabla u|^2 \geq \frac{(\Delta_f u)^2}{n+m} + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ric}_f^m(\nabla u, \nabla u)$

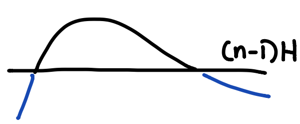
still holds

Note for $m = -n+1$, Wylie modified Bochner formula.

Integral Ricci

Notation: $\mathrm{Ric}_-^H = ((n-1)H - \rho(x))_+$ smallest eigenvalue of Ric

measures amt of Ricci below $(n-1)H$.



$$\|\mathrm{Ric}_-^H\|_{p,R} = \sup_{x \in M}\left(\int_{B(x,R)} (\mathrm{Ric}_-^H)^p\right)^{1/p}$$

$\psi = (\Delta r - \Delta_H r)_+$ — how much Laplace comparism failed.

$\mathrm{Ric} \geq (n-1)H \longleftrightarrow \|\mathrm{Ric}_-^H\|_p = 0$

Laplace comparism: $\|\mathrm{Ric}_-^H\|_p = 0$, $\Rightarrow \psi \equiv 0$.

Thm (Peterson-Wu 1997) M^n, $p > \frac{n}{2}$ $\left(H>0,\ r < \frac{\pi}{2\sqrt{H}}\right)$

$$\|\psi\|_{2p,R} \leq c(n,p)\left(\|\mathrm{Ric}_-^H\|_{p,R}^{1/2}\right)$$

note however that for $p \leq \frac{n}{2}$ the result is NOT true.

Volume comparison $\left(\frac{\mathrm{Vol}\,B(x,R)}{\mathrm{Vol}_H(B(R))}\right)^{1/2p} - \left(\frac{\mathrm{Vol}(B(x,r))}{\mathrm{Vol}_H(B(r))}\right)^{1/2p} \leq C(n,H,R)\left(\|\mathrm{Ric}^H\|_p\right)^{1/2}$

$r \to 0$ $\qquad\qquad\qquad\qquad$ $H=0,\ C(n,p)R^{1-n/2p}$

For application need $\left(⨍_{B(x,1)} \| Ric_-^H \|^p \right)^{1/p}$ small

no volume doubling without smallness

Thm (Dai-Wei-Zhang) M^n, $p > \frac{n}{2}$. If $\left(⨍_{B(x,1)} (Ric_-)^p \right)^{1/p} < \varepsilon(p,n)$,

then the Isoperimetric constant $Ist(B(x,r)) \leq \frac{c(n)\, r}{Vol(B(x,r))^{1/n}}$

$\sup_{\Omega} \left\{ \left(\frac{Vol(\Omega)^{1-1/n}}{Vol(\partial\Omega)} \right) \Big| \ \Omega \subset\subset B(x,r) \right\}$ ← the Dirichlet isometric constant

This constant can extend Cheeger-Colding-Naber work to collapsed case.

For integral curv have other vers.
Gallot, 1988
Yang 1992 (required $vol_M \geq v > 0$)

Tian-Zhang (performed in Kähler Ricci flow), application on integral curvature by extending the Cheeger-Colding result.