Aspects of Einstein Metrics on Four Manifolds II

$\overline{\mathcal{E}}_\lambda$ is compact for $\lambda > 0$. M^4 mfld.
L^2 completion (or GH completion if you prefer)

and $\overline{\mathcal{E}}_\lambda = \mathcal{E}_\lambda \cup \mathcal{E}_\lambda^{\text{orb sing}}$

"non collapsing sequence"

(M^4, g_i), $Ric_{g_i} = \lambda g_i$ fixed 4-mfld, metrics changing
$\lambda > 0$ fixed

Myer's Thm: $\operatorname{diam}_M g_i \leq D(\lambda)$, $\operatorname{vol}_{g_i} B(r) \geq v_0 r^4$.

Bishop Gromov: volume comparison thm.
If $\exists \Lambda < \infty$ s.t. $|Rm_{g_i}| \leq \Lambda \Rightarrow g_i \xrightarrow{C^\infty} g_\infty \in \mathcal{E}_\lambda$
C^∞ convergence.

Einstein eqns allow "boost" in regularity via elliptic regularity associated to the eqns

"converging to a pt in the interior of the moduli space"

Consider now when curvature blows up:
$\exists x_i \in M$, $|Rm_{g_i}|(x_i) = \|Rm_{g_i}\| \to \infty$
Rescale to bound the curvature (compact case)

$\tilde{g}_i = |Rm_{g_i}|(x_i) \cdot g_i$ ← focusing attention at behavior about x_i

Thus $|Rm|_{\tilde{g}_i}(x_i) = 1$ (renormalized), $\tilde{\lambda}_i = |Rm_{g_i}|^{-1} \lambda \to 0$.

$\Rightarrow$ (N, g_∞, x_∞) pointed G-H sense, C^∞ conv.
"$\lim x_i$"

$Ric_{g_\infty} \equiv 0$ on N, (N, g_∞) complete: Blow up limit "singularity model". using CGB below $\Rightarrow \int_N |R|^2 < \infty$ finite energy.

often Blow up limits becomes <u>simpler</u> than original setup.

Chern-Gauss Bonnet Thm
C-G-B in 4d.

$$\int_M |Rm|^2 - 4|Ric_0|^2 = 8\pi \chi(M).$$

<u>Einstein</u> $\int_M |Rm|^2 = 8\pi \chi(M) \leq \Lambda$
(scale invariant)

Observation of Berger:
(M^4, g) Einstein $\Rightarrow \chi(M) \geq 0 \Leftrightarrow g$ is flat.

ε-Regularity Theorem

take g Einstein. $\operatorname{vol} B(r) \geq v_0 r^4$

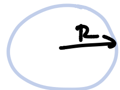
$|Rm|(x) \leq r^{-2} \left(\int_{B(r)} |Rm|^2 \right)^{1/2}$ provided $\int_{B(r)} |Rm|^2 < \varepsilon_0 - \varepsilon_0(\lambda, v_0)$.

similar version holds in higher dimensions

$r^2 |Rm|(x)$

- elliptic eqn
- uniform control of Sobolev constant
- Nash-Moser iteration

Consider $\int_N |R|^2 < \infty$, look at energy on arb. Large ball



Integral is small outside of that region.

ε regularity gives rate of decay via L^2 energy

So $g_\infty \to$ flat metric "fast"

(N, g_∞) is a flat ALE space

i.e. $N \times B(R_i) \simeq C(S^3/\Gamma) \quad \Gamma \leq SO(4)$

$\mathrm{Cone}(S^3) = (\mathbb{R}^4, \text{flat})$

Quantification

CBG for (N, g_∞)

$$\frac{1}{8\pi^2}\int_M |Rm|^2 = \underset{\geq 1}{\chi(M)} - \frac{1}{\underset{\geq 2}{|\Gamma|}} \geq \frac{1}{2}.$$

So can only have a bounded # of pts blowing up since "quantums" of energy are "eaten" by each pt.

RECENT WORK

- "Reverse Question" Given (V^4, g_∞) is there a smooth Einstein resolution? (einstein orbifold)

  that is, $\exists \underset{\uparrow \varepsilon}{(M^4, g_i)} \xrightarrow{GH} (V, g_\infty)$? (Related to gluing issues)

Via O. Biquard, answer is no, there is a Biquard obstruction

$$R = \left(\begin{array}{c|c} R^+ & Ric_0 \\ \hline Ric_0 & R^- \end{array}\right)$$

$\det(R^+_{g_\infty}(q)) = 0$ if $\Gamma = \mathbb{Z}_2$. q any singular pt of (V, g_∞)

Example $K3 := \underset{\substack{\uparrow \\ \text{flat}}}{T^4/\mathbb{Z}_2} \# \overline{TS^2}$ $\nwarrow N$

consider seq of metrics $\to \infty$, curv blows up, get (N, g_∞) w/ $g_\infty =$ Eguchi-Hanson metric

$$\frac{1}{1-(\frac{a}{r})^4} dr^2 + \left(1 - \left(\frac{a}{r}\right)^4\right) r^2 \theta_1^2 + r^2(\theta_2^2 + \theta_3^2)$$

$r \geq a$ with $(\theta_1, \theta_2, \theta_3)$ orth. framing for $\mathbb{RP}^3 \simeq S^3/\mathbb{Z}_2$

$\frac{T^4}{\mathbb{Z}_2} \# \, kTS^2 \# (16-k)\overline{TS^3}$

Does not admit KR flat metric

$|Ric|_{g_\varepsilon} \leq \varepsilon, \quad \forall \varepsilon > 0.$

Recently addressed in paper by Brendle & Kapouleus.
"There does not exist Ricci flat metric "too near" g_ε".

Leads to new ancient Ricci solitons.

A. Naber — Cheeger - codim 4 sing set
— R. Zhang - ε regularity thm in collapsing setting

T. Colding & W. Minicozzi - uniqueness of tangent cones.

II Existence $\Delta u = 0 \overset{M \text{ complete}}{\Longrightarrow} u$ constant.

mflds w/ boundary ∂M. $\longrightarrow$ $\mathcal{E}^{m,\alpha} = \dfrac{\mathbb{E}^{m,\alpha}}{\mathrm{Diff}^{m,\alpha}_{\partial M}}$ (fix ∂M)
M^n w/ $\partial M \neq 0$.

<u>Thm</u> $\mathcal{E}^{m,\alpha}$ is a smooth ∞-dim Banach mfld.

Boundary data?
$\Pi_D : \mathcal{E}^{m,\alpha} \longrightarrow \mathrm{Met}^{m,\alpha}(\partial M)$
$g \longrightarrow \gamma = g/_{\partial M}$

Simplest case
$n=3$, $\lambda=0$, $M=B^3$, $g=$ flat metric
Q: given γ on S^2, $\exists ? g$ on B^3 so that you have an isometric immersion?
classical problem.

Dirichlet data are <u>not</u> elliptic. Why? Gauss eqns are the obstruction.

$$|A|^2 - H^2 + R_g = (n-1)\lambda.$$

If D-data is elliptic, $\gamma \in C^{m,\alpha} \Rightarrow A \in C^{m-1,\alpha}(\partial M) \longleftarrow \not\Longrightarrow$
elliptic ineq $\Rightarrow R_\gamma \in C^{m-1,\alpha}(\partial M)$