

MAXIMAL SYMMETRY RANK CONJECTURE FOR NONNEGATIVE CURVATURE

For class of manifolds M^n , $\pi_1 = 0$, closed, we still cannot distinguish between $sec > 0$, $sec \geq 0$. One way to deal with this lack of knowledge is to introduce symmetries.

Consider Abelian symmetries

Defⁿ: The symmetry rank of mfd $symrk(M) = rk[Isom(M)]$ isometry group

Note M closed $\Rightarrow Isom(M)$ is ∞ Lie group $rk \rightarrow$ max iml torus in that Lie group.
Consider the rank of torii acting

Theorem (Maximal Symmetry Rank Theorem) (Grove, S-)

Let T^k act isometrically and effectively on M^n , $sec > 0$, closed. Then

(i) $k \leq \lfloor \frac{n+1}{2} \rfloor$

(ii) in the case of equality, M^n is diffeomorphic to S^n , \mathbb{R}^n , $L_{p,q}$ or $\mathbb{C}P^m$ ($2m=n$).
(lens space)

Roy: AMSR: $k = \lfloor \frac{n-1}{2} \rfloor$ dim 5: S^5 & obtained topological restrictions in higher dims
"1/4" rank

Wilking: $k \geq \frac{n}{4} + 1$, used connectivity lemma to show either M^n is homeomorphic to S^n , $\mathbb{H}P^k$ or topologically homeomorphically equivalent to $\mathbb{C}P^m$.

Fang + Rong AMSR: $n \geq 8$ home to S^n , $\mathbb{H}P^k$, or $\mathbb{C}P^m$.
almost maximal symmetry rank

Rmk: AMSR in dim 8, 9 is "1/4" rank, that is, Wilking's result was extended.

OPEN PROBLEM: classification of AMSR for dim $n = 6$ & 7 .

$$T^2 \curvearrowright M^6, T^3 \curvearrowright M^7$$

In dim 4: $T^1 \curvearrowright M^4, \pi_1 = 0 \Rightarrow \chi(M^4) \geq 2 = \chi(\text{Fix}(M; S^1))$
comes from Poincare duality, you can't say that for higher dim
in presence of a circle action where this denotes fixed pts.
 $sec > 0: S^4, \mathbb{C}P^2$ $sec \geq 0: S^4, \mathbb{C}P^2, S^2 * S^2, \mathbb{C}P^2 \# \pm \mathbb{C}P^2$

Similar conjecture for nonnegative sectional curvature

Conjecture (Maximal Symmetry rank)

Let $T^k \curvearrowright M^n, \pi_1 = 0$, closed, $sec \geq 0$. Then

version 1

i) $k \geq \lfloor \frac{2n}{3} \rfloor$

version 2

ii) In the case of equality and for $n = 3m$ ($n \equiv 0 \pmod{3}$)

M^n is diffeo to $S^3 \times \dots \times S^3$

(Replace with more general conjecture) M^n is equivalently m times

diffo to $N = \prod_{i=1}^r S^{2n+1} \times \prod_{i=1}^r S^{2n+1} \times \prod_{i=1}^r S^{2n}$, with $r = 2 \lfloor \frac{2n}{3} \rfloor - n$, or in the case $n \not\equiv 0 \pmod{3}$, the quotient by a free linear action of a torus of $2n \pmod{3}$ of N

Note, this discussion forward is joint work with Christine Escher

Q: what do we know about this conjecture?

- True in $dim \leq 6$
- The classification results in dim's 2 + 3 is well known
- For dim's 4 & 5 + 6, classification up to diffeo Galaz-Garcia & S-

- 4, 5, 6 improved to equivariant diffeo by Galaz-Garcia & Kerin.
- (i) is true for $\dim \leq 9$ Galaz-Garcia & S-

Bott Conj Let M be closed $\pi_1=0, \text{sec} \geq 0$ Riem mfd. Then M is rationally elliptic.

$$\dim_{\mathbb{Q}} H^*(X; \mathbb{Q}) < \infty \ \& \ \dim_{\mathbb{Q}} (\pi_k(x) \otimes \mathbb{Q}) < \infty.$$

Examples of pos & nonneg curv mflds are all rational

Thm (Galaz-Garcia, Kerin, Ralesdu)

Let T^k act smoothly & effectively on M^n a $\pi_1=0$ closed and rationally elliptic. Then

(i) $k \leq \lfloor \frac{2n}{3} \rfloor$ ← expected.

(ii) The free rank of the action of the action $\leq \lfloor \frac{n}{3} \rfloor$, where the free rank corresponds to the rank of the largest subtorus that can act almost freely

Prop (Escher, S- / GEKR others?)

Let $T^k \curvearrowright X^n$, a closed Alexandrov space with a lower bound, & $k \geq \lfloor \frac{n+1}{2} \rfloor$. Then the free rank of the T^k action is $\geq 2k-n$.

Theorem (Escher, S-)

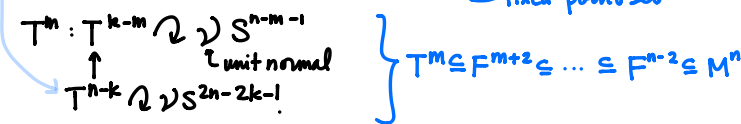
Let $T^k \curvearrowright M^n, \pi_1=0, \text{sec} \geq 0$, closed rationally elliptic. Then the Maximal Rank Conjecture holds.

Theorem B (Escher, S-) Let $T^k \curvearrowright M^n$, closed $\pi_1=0, \text{sec} \geq 0$.

Assume that when $k = \lfloor \frac{2n}{3} \rfloor$ then the action is maximal or almost maximal. Then part (ii) of the maximal symmetry rank conjecture holds.

maximal action: $2k+n=m \leftarrow \text{dim of smallest orbit.}$

almost maximal: $2k+n=m-1 \rightarrow T^m \subseteq F^{m+1} \subseteq F^{n-2} \subseteq M^n$
 — fixed point set



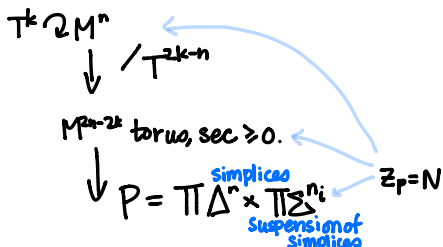
Theorem (Escher, S-) Let $T^k \curvearrowright M^n, 7 \leq n \leq 9, M^n$ closed, $\pi_1=0, \text{sec} \geq 0 \Rightarrow$ the MSR conjecture holds.

Cor D (Escher, S-) same hypothesis, ^{$\dim \leq k$} Then part (i) of the MSR conjecture is true.

A torus manifold is M^{2n} , closed, orientable with a T^n action $\exists M^{T^n} \neq \emptyset$

Theorem (Wiener), Let M^n be a $\pi_1=0, \text{sec} \geq 0$ torus manifold. Then M^n is equivariantly diffeomorphic to a free linear quotient by a torus of N .

$$N = \prod_{i=1}^r S^{2m_i+1} \times \prod_{i=r+1}^n S^{2m_i}$$



2 cases $k = \# \text{ faces}(P),$
 $k < \# \text{ faces}(P)$

Open questions & Problems

1) Classify almost MSR.

Results: Dim's 2, 3.

Dim 4: $\text{sec} > 0$ HK, $\text{sec} \geq 0$ K/S^1 T^1

Dim 5: T^2 action: Rong $\text{sec} > 0$, $\text{sec} \geq 0$ (GGS)

Dim 6: T^3 action: MSR $\text{sec} > 0$, $\text{sec} \geq 0$, $S^3 \times S^3$ or torus mfld

2) For cohomogeneity m torus actions, $m \geq 4$ can one show that $\text{Fix}(M; S^1) \neq \emptyset$ for some circle?
(with large torus already acting on it).