Maximal Symmetry Rank Conjecture for Nonnegative Curvature

For class of manifolds M^n, $\pi_1 = 0$, closed, we still cannot distinguish between
$\sec > 0$, $\sec \geq 0$. One way to deal with this lack of knowledge is to introduce symmetries.
Consider Abelian symmetries
Def$^{\underline{n}}$ The symmetry rank of mfld $\mathrm{symrk}(M) = \mathrm{rk}[\underbrace{\mathrm{Isom}(M)}]$ isometry group

Note M closed $\Rightarrow$ $\mathrm{Isom}(M)$ is cmpt Lie group   rk $\to$ maximl torus in that lie group.
Consider the rank of torii acting

Theorem (Maximal Symmetry Rank Theorem) (Grove, S–)
Let T^k act isometrically and effectively on M^n, $\sec > 0$, closed. Then
(i) $k \leq \lfloor \frac{n+1}{2} \rfloor$
(ii) in the case of equality, M^n is diffeomorphic to S^n, $\mathbb{R}^n$, $L_{p,q}$ (lens space) or $\mathbb{C}P^m$ $(2m = n)$.

Roy: AMSR ("1/4 rank"): $k = \lfloor \frac{n-1}{2} \rfloor$ dim 5: S^5 & obtained topological restrictions in higher dim's

Wilking: $k \geq \frac{n}{4} + 1$, used connectivity lemma to show either M^n is homeomorphic to S^n, $\mathbb{H}P^k$ or topologically homeomorphically equivalent to $\mathbb{C}P^m$.

Fang + Rong AMSR (almost maximal symmetry rank): $n \geq 8$ homo to S^n, $\mathbb{H}P^k$, or $\mathbb{C}P^m$.

Rmk: AMSR in dim 8, 9 is "1/4" rank, that is, Wilking's result was extended.
OPEN PROBLEM: classification of AMSR for dim's 6 & 7.

$T^2 \circlearrowright M^6$, $T^3 \circlearrowright M^7$.

In dim 4: $T^1 \circlearrowright M^4$, $\pi_1 = 0 \Rightarrow \chi(M^4) \geq 2 = \chi(\mathrm{Fix}(M; S^1))$ comes from Poincare duality, you can't say that for higher dims
(in presence of a circle action where this denotes fixed pts.)

$\sec > 0$: S^4, $\mathbb{C}P^2$ $\sec \geq 0$: S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$

Similar conjecture for nonnegative sectional curvature

Conjecture (Maximal Symmetry rank)
Let $T^k \circlearrowright M^n$, $\pi_1 = 0$, closed, $\sec \geq 0$. Then

version 1
i) $k \geq \lfloor \frac{2n}{3} \rfloor$

version 2
ii) In the cases of equality and for $n = 3m$ $(n = 0 \bmod 3)$
M^n is diffeo to $\underbrace{S^3 \times \cdots \times S^3}_{m \text{ times}}$

(Replace with more general conjecture) M^n is equivalently diffeo to $N = \prod_{1 \leq r} S^{2n+1} \times \prod_{i > r} S^{2n+1} \times \prod_{i > r} S^{2n}$, with $r = 2\lfloor \frac{2n}{3} \rfloor - n$, or in the case $n \not\equiv 0 \bmod 3$, the quotient of N by a free linear action of a torus of $2n \bmod 3$

Note, this discussion forward is joint work with Christine Escher

Q: what do we know about this conjecture?

- True in dim ≤ 6
- The classification results in dim's 2 + 3 is well known
- For dim's 4 & 5 + 6, classification up to diffeo Galaz-Garcia & S–

- 4,5,6 improved to equivariant diffeo by Galaz-Garcia & Kerin.
- (i) is true for dims ≤ 9 Galaz-Garcia & S-

<u>Bott Conj</u> Let M be closed $\pi_1 = 0$, $\sec \geq 0$ Riem mfld. Then M is ratially elliptic.

$$\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) < \infty \ \& \ \dim_{\mathbb{Q}}(\pi_k(x) \otimes \mathbb{Q}) < \infty .$$

Examples of pos & nonneg curv mflds are all rational

Thm (Galaz-García, Kerin, Ralesdu)

Let T^k act smoothly & effectively on M^n a $\pi_1 = 0$ closed and rationally elliptic. Then

(i) $k \leq \lfloor \frac{2n}{3} \rfloor$ ← expected.

(ii) The free rank of the action of the action $\leq \lfloor \frac{n}{3} \rfloor$, where the <u>free rank</u> corresponds to the rank of the largest subtorus that can act almost freely

Prop (Escher, S- / GKR' others?)

Let $T^k \circlearrowright X^n$, a closed Alexandrov space with a lower bound, & $k \geq \lfloor \frac{n+1}{2} \rfloor$. Then the free rank of the T^k action is $\geq 2k - n$.

Theorem (Escher, S-)

Let $T^k \circlearrowright M^n$, $\pi_1 = 0$ $\sec \geq 0$, closed rationally elliptic. Then the Maximal Rank Conjecture holds.

Theorem B (Escher, S-) Let $T^k \circlearrowright M^n$, closed $\pi_1 = 0$, $\sec \geq 0$.

Assume that when $k = \lfloor \frac{2n}{3} \rfloor$ then the the action is maximal or almost maximal. Then part (ii) of the maximal symmetry rank conjecture holds.

<u>maximal action</u>: $2k + n = m$ ← dim of smallest orbit.

<u>almost maximal</u>: $2k + n = m - 1$ $\longrightarrow T^m \subseteq F^{m+1} \subseteq \dots \subseteq F^{n-2} \subseteq M^n$

— fixed point set

$T^m : T^{k-m} \circlearrowright \circlearrowright S^{n-m-1}$ ← unit normal

$\uparrow$

$T^{n-k} \circlearrowright \circlearrowright S^{2n-2k-1}$.

$\Big\} \ T^m \subseteq F^{m+2} \subseteq \dots \subseteq F^{n-2} \subseteq M^n$

Theorem (Escher, S-) Let $T^k \circlearrowright M^n$, $7 \leq n \leq 9$. M^n closed, $\pi_1 = 0$, $\sec \geq 0 \Rightarrow$ the MSR conjecture holds.

<u>Cor D</u> (Escher, S-) same hypothesis & $\dim \leq k$. Then part (i) of the MSR conjecture is true.

A torus manifold is M^{2n}, closed, orientable with a T^n action $\ni M^{T^n} \neq \emptyset$

<u>Theorem</u> (Wieneler), Let M^n be a $\pi_1 = 0$, $\sec \geq 0$ torus manifold. Then M^n is equivariantly diffeomorphic to a free linear quotient by a torus of N.

$$N = \prod_{i \leq r} S^{2n_i + 1} \times \prod_{i < r} S^{2n_i}$$

$T^k \circlearrowright M^n$

$\downarrow \ / T^{2k-n}$

M^{2n-2k} torus, $\sec \geq 0$.

$\downarrow$

$P = \prod \Delta^n \times \prod \Sigma^{n_i}$ (simplices; suspension of simplices)

$Z_P = N$

2 cases $k = \#\text{faces}(P)$,
$k < \#\text{faces}(P)$

Open questions & Problems

1) Classify almost MSR,

Results : Dim's 2, 3.
Dim 4 : sec > 0 HK, sec ≥ 0 K/sy T^1
Dim 5 : <u>T^2 action</u> : Rong sec > 0, sec ≥ 0 (GGS)
Dim 6 : <u>T^3 action</u> : MSR sec > 0, sec ≥ 0, $S^3 \times S^3$ or torus mfld

2) For cohomogeneity m torus actions, $m \geq 4$ can one show that $\mathrm{Fix}(M; S^1) \neq \emptyset$ for some circle?
(with Large torus already acting on it).