MINIMAL SUBMANIFOLDS AND LOWER CURVATURE

Yesterday: focused on codim 1 case. Now we extend.

2 dimnl case → interesting methods via complex geometry

2 dimn'L surface — the induced metric on the surface induces isothermal coordinates

<u>k=2</u> $u : \Sigma^2 \to M^n$, $V \in \Gamma(NM)$ normal bundle, assume M orientable. x^1, x^2, $z := x^1 + i x^2$ isothermal coords

$$\delta^2 \Sigma(V,V) = \int_\Sigma \sum_{i=1}^{2} |\nabla^\perp_{x^i} V|^2 - \underbrace{\sum_i |\nabla^T_{x_i} V|^2}_{\text{2nd fund form}} - \sum_i R(u_{x^i}, V, u_{x^i}, V)]\, dx^1 dx^2 \quad (\star)$$

coords not dependent on metric

$V = V_1 + i V_2$ ← can encode pair of variations as complex variation.

each term has conformormal invariance.

We rewrite $(\star)$ in a better way to understand its geometric meaning via complex coords

so $\delta^2\Sigma(V,V) = \int_\Sigma [\,|\nabla^\perp_z V|^2 + |\nabla^\perp_{\bar z} V|^2 - |\nabla^\perp_z V|^2 - |\nabla^\perp_{\bar z} V|^2 - R(u_z, V, u_{\bar z}, \bar V) - R(u_{\bar z}, V, u_z, \bar V)\,]\, dx^1 d^2$

equal up to curv / bdry terms

IBP to get rid of z derivatives
(Σ either cmpt or $V=0$ on $\partial\Sigma$ so no bdry term)

$$\delta^2\Sigma(V,V) = \int_\Sigma [\,\underset{\overset{\shortparallel}{0}}{|\nabla^\perp_{\bar z} V|^2} - |\nabla^T_z V|^2 - \underbrace{R(u_z, V, u_{\bar z}, \bar V)}_{K_{\{u_z, V\}}}\,]\, dx^1 dx^2$$

Cauchy Riem.

$u_{x^1} - i u_{x^2}$

2nd var: we can deal with these terms

note $|u_x|^2 = |u_y|^2$, $\langle u_x, u_y \rangle = 0$,
$\boxed{\text{so } (u_z, u_z) = 0}$

$R(\cdot,\cdot,\cdot,\cdot)$ complex linear $\quad \pi = \{x, y\}$, $K_\pi = R(x, y, \bar x, \bar y)$.

(ideas from paper
1988 Micallef & Moore)

Take W complex vector, $(\cdot,\cdot)$ complex linear extension of g, $\langle\cdot,\cdot\rangle$ hermitian.
W is <u>isotopic</u> if $(W,W) = 0$, $W = W_1 + i W_2$
$\Leftrightarrow |W_1|^2 = |W_2|^2$, $\langle W_1, W_2 \rangle = 0$

Def$^{\underline{n}}$ M has <u>positive isotropic curvature</u> (PIC) if $K_\pi > 0$ for all isotropic π.

Thm (Micallef & Moore) Geometric Thm: If $u: S^2 \to M^n$ ($n \geq 4$) is minimal (harmonic) (i.e. conformal parametrization)
& M is PIC then $\mathrm{Ind}(u(S^2)) \geq [\,|\frac{n-2}{2}|\,]$

Pf uses special facts about hol'c bundle over 2-sphere.
In 2-D have integrability on normal bundle

Idea: Construct sufficiently many $V \in \Gamma(N_{\mathbb{C}}\Sigma)$ with $\nabla^\perp_{\bar z} V = 0$ & $(V,V) = 0$.

Combine w/ topological results to obtain

Cor: Topological sphere thm for ptwise quarter pinching

Note: "ptwise 1/4 pinching" means $\forall p \in M$ with $K_M > 0$, $\pi_1(M) = \{1\}$, have $\frac{\max K_p(\pi)}{\max K_p(\pi)} < 4$.

<u>Brendle-S</u> PIC is preserved under Ricci flow.

2 conditions related to PIC: $\quad$ PIC $\supseteq \underset{\text{Brendle}}{M \times \mathbb{R}}$ PIC $\supseteq M \times \mathbb{R}^2$ PIC $\geq$ 1/4-pinching

M is a spherical space form

Q: what about PIC?

Thm (Micallef-Wang) If M_1, M_2 are PIC then $M_1 \# M_2$ has PIC metric

M_1 M_2

S^n/Γ, $S^{n-1} \times S^1$. (remember, $n \geq 4$)

get "handle-bodies". Free gps

PIC cnj: If M (compact) is PIC then M is finitely covered by $(S^{n-1} \times S^1) \# \cdots \# (S^{n-1} \times S)$

In dim 4 the Ricci flow for PIC is special.

n=4 R. Hamilton, B. Chen, X. Zhu.
PIC conjecture holds.

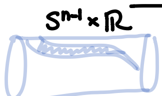
Next: π_1-PIC conjecture If M is PIC $\Rightarrow$ $\pi_1(M)$ is virtualy free.
(there is free subgp in π_1 which has finite index).

Thm (Gadgil-Seshadri) If M is PIC & has free π_1 then M is homeomorphic to
$(S^{n-1} \times S^1) \# \cdots \# (S^{n-1} \times S^1)$

There is a minimal surface approach to this.

Geometric Question: $\tilde{M}$ universal cover.
M has finite fill radius if $\forall \Gamma \subseteq \tilde{M}$, $\exists \Sigma$ disk $\partial\Sigma = \Gamma$ & $\Sigma \subseteq \eta_c(\Gamma)$
distance nbhd of Γ

$S^{n-1} \times \mathbb{R}$



Thm (M. Ramachandran, J. Wolfrom 2010). If M has finite fill radius then $\pi_1(M)$ is virtually free.

$\Gamma \subseteq \tilde{M}$, $\exists$ Σ least area disk $\partial\Sigma = \Gamma$.
Hope: $\Sigma \subseteq \eta_c(\Gamma)$ for some c fixed.

True when $\dim M = n = 3$, $R_M > \kappa$. A Bonnet type thm occurs
$\Sigma^2 \subseteq M$ stable $\Rightarrow \operatorname{diam}(\Sigma) \leq 2\pi/\sqrt{\kappa}$

$L \leq 2\pi/\sqrt{\kappa}$

Γ

Minimal surface conjecture:
M, $IC > \kappa$ & Σ stable minimal disk,
then $\forall p \in \Sigma$, $d(p, \partial\Sigma) \leq c/\sqrt{\kappa}$

Rmks on Conj.
2nd Variation: $\kappa \int_\Sigma |V|^2 d\mu \leq \int_\Sigma |\nabla^{\perp}_{\bar{z}} V|^2 d\mu$ one way to get rids of term is to construct hol'c iso. sections

If disk is quite large, forces eigenvalue to be small.
Idea: V hol'c, isotropic, slow growth.

Thm (A. Fraser, 2003) If M is PIC then $\pi_1(M)$ contains no copy of $\mathbb{Z} \times \mathbb{Z}$.

So torsion subgps can be represented by minimal torii

Stability for T^2 $\quad S^{n-1}/\Gamma \times S^1 \to \sigma_i \times S^1$.

The Theorem $\Rightarrow$'s If $u: T^2 \to M$ PIC. minimal then $u: (kT^2) \to M$ unstable for k large.

Notion: stable homology, from alg. geom. If you have a homology class representable by a calibrated/complex submanifold then every covering is minimizing

Frasers thm says: although you may have stable torus, for sufficienty high covering get instability.

Basic Idea: Construct approx holom isotropes V on kT^2.

$$\int_{k\Sigma} |\nabla_{\frac{\partial}{\partial \bar{z}}} V|^2 \leq \varepsilon(k) \int_{k\Sigma} |V|^2.$$

analog. to Gromov & Lawson ideas

Takes, for k large, the large torus will have a contracting map to S^2, then can take a line bundle on S^2 and pull it back, tensor it to a complexified normal bundle, this has a holomorphic section.
Use hol'c bundle with fact that map is distance decreasing, you can make it almost hol'c in this sense.

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Geometric Result for $\cancel{\Sigma^2}\ M^3$, $R \geq 1$:

(SY)

$\tilde{M}$ has bounded fill radius

Γ curve. $\exists\, D$ $\partial D = \Gamma$.

$$D \subseteq N_{\frac{2\pi}{\sqrt{3}}}(\Gamma)$$

(*)

Idea: $u_0 > 0$ $\quad Lu_0 \leq 0$.

Weighted arclength $L_{u_0}(\sigma) = \int_\sigma u_0\, ds$.

Satisfies Bonnet-type property if $R \geq 1$.

with dist measured in g.

$k = 2$: $\quad \Sigma^2 \hookrightarrow M^n$.

Complex form: $\quad (\, , \,)$ complex linear

$\langle \, , \, \rangle$ hermitian pairing

$$\begin{array}{c} N_{\mathbb{C}}\Sigma \\ \downarrow \\ (\Sigma, z) \end{array}$$

$R(\cdot, \cdot, \cdot, \cdot)$ complex linear

Index form: $W \in \Gamma(N_{\mathbb{C}}\Sigma)$.

$$I(W,W) = \int_\Sigma \left[|D_z^\perp W|^2 - R(W, u_z, \overline{W}, u_{\bar{z}}) - |(\nabla_z W)^T|^2 \right]$$

<u>Micallef-Moore</u>: $(u_z, u_z) = 0$

Isotropic

<u>Def</u>: M has <u>positive isotropic curvature</u> (PIC) if

$K(\pi) > 0 \quad \forall \pi$ isotropic 2 dim'l complex.

$S^2 \hookrightarrow M^n$ PIC

ind $\geq \left[\frac{n-2}{2}\right]$

$\pi_1(M) = \{1\}$ PIC $\Longrightarrow$ M homeo to S^n.

<u>Cor</u>: $K > 0 \quad \forall P \quad \frac{\max K(P)}{\min K(P)} < 4$

$\pi_1 = \{1\}$

$\Longrightarrow M \approx S^n$.

5 <u>Brendle-S</u>: Cor improved to differ without simply connected.

{Ricci Flow} $\supseteq M\times\mathbb{R} \supseteq M\times\mathbb{R}^2$

Sph. Sph.

<u>Topological Classificat. of PIC</u>:

$n=4$, R. Hamilton, B. Chen - X. Zhu

<u>$n \geq 5$</u> ?

~~Non-Simply Connected A. Fraser 2003 $\pi_1 \not\supset \mathbb{Z}\oplus\mathbb{Z}$~~.

Route to topological classification:

Conjecture: π_1 is virtually free

Thm: (M. Ramachandran, J. Wolfson 2010)
Conjecture follows from $\tilde{M}$ has bounded fill radius

Thm: (S. Gadgil, H. Seshadri 2009) If
M is PIC & $\pi_1(M)$ is a free group then
M is homeomorphic to connected sum
of k copies of $S^1 \times S^{n-1}$.

, dist, stable
Conjecture: $(\Sigma, \partial\Sigma) \subseteq M$ $K(\pi) \geq 1$.

$\Longrightarrow \Sigma \subseteq N_c(\partial\Sigma)$

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