Fully Nonlinear Flows with Surgery

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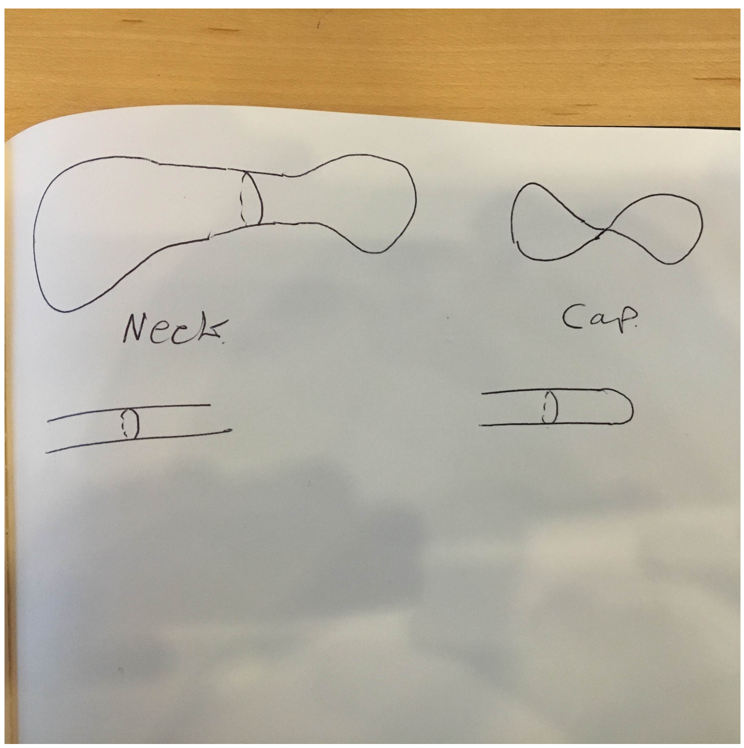
This is joint work with G. Huisken. We will look at n-dimensional hypersurfaces $M^n \subseteq N^{n+1}$ where we assume that M is smooth, closed, and embedded. To fix notation we will denote the second fundamental form as h_{ij} with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. We will be interested in parabolic flows.

Definition 1 *M is 2-convex if and only if $\lambda_1 + \lambda_2 > 0$.*

One natural way to evolve a hypersurface is by curvature flow. The speed will be given by $-H\nu$ where $H = \lambda_1 + \ldots + \lambda_n$.

Theorem 1 (Huisken-Sinestrari) *Let $n \geq 3$. Assume $M_0^n \subset \mathbb{R}^{n+1}$ is 2-convex, embedded, closed, and evolve M_0 by the mean curvature flow. Then there exists a mean curvature flow with finitely many surgeries which become extinct in finite time.*



The question now becomes how we perform the surgery. Either via a *neck* modeled by a cylinder, or a *cap*. The case for $n = 2$ was proven analagously by Brendle and Huisken, as well as by Haslhofer-Kleiner. What's interesting about the lower dimensional case is that this framework extends over to the case of hypersurfaces in a Riemannian manifold provided either

- the dimension of the hypersurface is 2, or
- if $n \geq 3$ the space is symmetric (i.e. $\overline{\nabla Rm} = 0$).

If neither of these cases hold then 2-convexity is not preserved in higher dimensions in non-symmetric spaces. To get around this problem, we would like to use a flow that preserves 2-convexity in the Riemannian setting. There are a couple of different flows that can satisfy this, but they are all of the same basic type. The first mean curvature flow can have speed $-G\nu$ where

$$G = \left(\sum_{i<j} \frac{1}{\lambda_i + \lambda_j} \right)^{-1}.$$

In any Riemannian manifold this type of flow will preserve 2-convexity. The issue however is that this flow is fully nonlinear. There are however some basic algebraic properties for 2-convex. Recalling that $\lambda_1 + \lambda_2 > 0$ we have

1. $0 < \frac{\partial G}{\partial \lambda_i} \leq C(n)$,
2. $G \leq C(n)H$,
3. G is strictly concave (the Hessian of G gives a quadratic form).

If we look at the evolution of G we have

$$\frac{\partial G}{\partial t} = \frac{\partial G}{\partial h_{ij}} (D_i D_j G + h_{ik} h_{jk} G + \overline{R}_{i\nu j\nu} G)$$

Note that the infimum of G is bounded below as long as t is bounded. If we additionally impose the assumption that the ambient curvature $\overline{R}_{1313} + \overline{R}_{2323} \geq 0$ then you can demonstrate that $\inf G$ blows up in finite time. It is this last property that gives us finite time extinction.

Lemma 1 *$\frac{G}{H}$ is uniformly bounded below for bounded time intervals. Furthermore this implies that the flow is uniformly parabolic.*

Proof: This can be easily shown by the Maximum principle. □

Theorem 2 (Brendle, Huisken) *Let M_t^n be a flow with speed G.*

1. ***Convexity Estimate**: For every $\delta > 0$ there exists $C = C(M_0, N, T, \delta)$ such that $\lambda_1 \geq -\delta G - C$.*
2. ***Cylindrical Estimate**: For every $\eta > 0$ there exists $\delta > 0$ and C such that if $\lambda_1 \leq \delta G$ then this implies that $\lambda_n - \lambda_2 \leq \eta G$.*
3. ***Inscribed Radius Bound**: (Established by Langford-Andrews-McCoy) The inscribed radius is $\geq \frac{\alpha}{G}$ where $\alpha = \alpha(M_0, N, T)$.*
4. ***Curvature Derivative Estimate**: There exists constant Λ such that $\alpha^2 G^{-2} |\nabla h| + \alpha^3 G^{-3} |\nabla^2 h| \leq \Lambda$.*

Corollary 1 *There exists a flow with finitely many surgeries in bounded time intervals.*

If N satisfies that condition that $\overline{R}_{1313} + \overline{R}_{2323} \geq 0$, then the flow becomes extinct in finite time.

We shall now focus on the estimates of this theorem.

Proof: Consider the case where $N = \mathbb{R}^{n+1}$. Note that $\sup \frac{H}{G}$ is monotone decreasing, but this monotonicity is strict unless M is a cylinder or a sphere. This can be shown by using Stampacchia iteration, which yields that inequality

$$H \leq \left(\frac{(n-1)^2(n+2)}{4}+\delta\right)G + C(\delta, T, M_0).$$

Note that on $\mathbb{S}^{n-1}\times\mathbb{R}$ we have that $H = \frac{(n-1)^2(n+2)}{4}G$. In the blow-up limit we have that $H \leq \frac{(n-1)^2(n+2)}{4}G$. Algebraically we have $\lambda_1 \geq 0$ and $\lambda_1 = 0$ only if $\lambda_i = 0$ for all $i \in \{2,\dots,n\}$. Using this inequality, we get both the convexity and cylindrical estimates. We now focus on the curvature derivative estimate (the inscribed radius bound follows from the work done by Langford-Andrews-McCoy which is straightforward). This result was shown by Brendle-White using GMT and monotonicity, along with the result by Haslhofer-Kleiner where we take $H > 0$, and Huisken-Sinestravi using the maximum principle (this result requires 2-convexity). We need another result in order to prove this last part.

Theorem 3 (Splitting Theorem) *Suppose we have a smooth blow-up limit. Then this implies that $\lambda_1 \geq 0$ everywhere. Furthermore, if $\lambda_1 = 0$ somewhere on the limit, this implies that the flow is a family of shrinking round cylinders.*

Theorem 4 *Suppose we have a flow $M_t = \partial\Omega_t$ with $t \in [-r^2, 0]$ and $B_r(p) \subset \Omega_t$ for all t, and additionally that $\langle x-p, \nu(x)\rangle \geq 10^{-3}\nu$ with $G \geq \beta H$ for all $x \in M_t \cap B_{2r}(p)$, then this implies that*

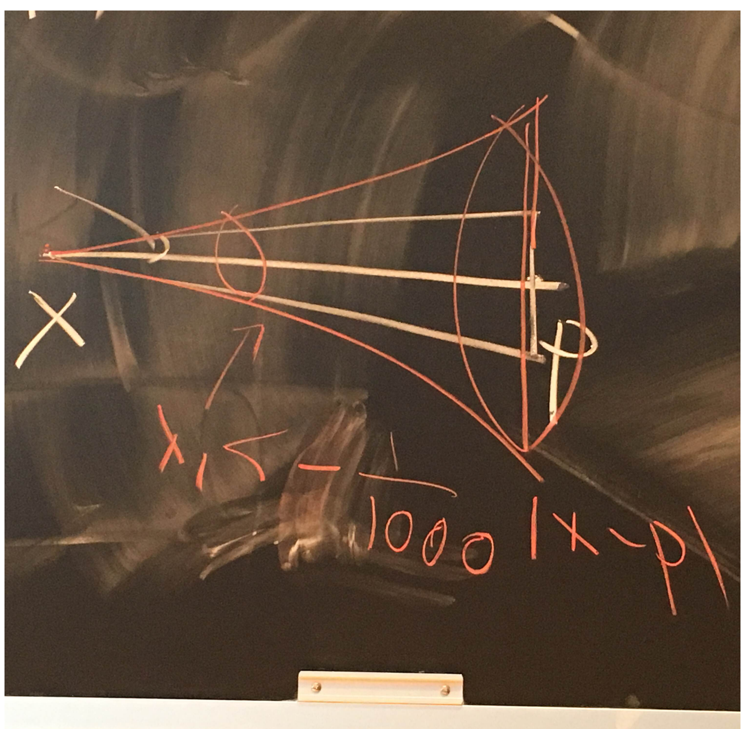
$$r^2|\nabla h| + r^3|\nabla^2 h| \leq \Lambda$$

for all $x \in M_t \cap B_{\frac{4r}{3}}(p)$.

We now come back to the proof of curvature derivative estimate. We argue this by contradiction. Suppose there exists (x_k, t_k) such that $F(x_k, t_k) \to \infty$ and $\alpha^2 G|\nabla h| + \alpha^3 G^{-3}|\nabla^2 h| > \Lambda$ at (x_k, t_k). Pick points $(\overline{x}_k, \overline{t}_k)$ such that $\overline{t}_k < t_k$. Then we have that $G(\overline{x}_k, \overline{t}_k) > G(x_k, t_k) \to \infty$, and therefore $\alpha^2 G^{-2}|\nabla h| + \alpha^3|\nabla^2 h| > \Lambda$ and therefore

$$\alpha^2 G^{-2}|\nabla h| + \alpha^3|\nabla^2 h| \leq \Lambda$$

for all (x,t) such that $t \leq \overline{t}_k$ and $G(x,t) \geq 2G(\overline{x}_k, \overline{t}_k)$. Given points p, x we will let $C_{p,x}$ be the pseudo cone with vertex at x. Here we will have that $\lambda_1 < -\frac{1}{1000}|x-p|$.



Consider the inscribed ball at $(\overline{x}_k, \overline{t}_k)$ with center at $p_k \in \mathbb{R}^{n+1}$ and $r_k = \alpha G(\overline{x}_k, \overline{t}_k)^{-1}$, then we have two possibilities:

- $C_{p_k,x} \subset \Omega_t$, for all $t \in [\overline{t}_k - r_k^2, \overline{t}_k]$ and for all $x \in \Omega_t \cap B_{2r_k}(p_k)$ implying that the space is star-shaped and therefore $\alpha^2 G^{-2}|\nabla h| + \alpha^3 G^{-3}|\nabla^2 h| \le \Lambda$ at $(\overline{x}_k, \overline{t}_k)$ which contradicts our assumption.

- the psuedo cone is not always contained in Ω_t. In this case, we look at a borderline situation where (for some $\tilde{t}_k \in [\overline{t}_k - r_k^2, \overline{t}_k]$) a pseudo cone is contained in $\Omega_{\tilde{t}_k}$ and touches the boundary $M_{\tilde{t}_k} = \partial\Omega_{\tilde{t}_k}$ from the insite at some point y_k. At the point y_k, we have $\lambda_1(y_k, \tilde{t}_k) < -\frac{1}{1000}\frac{1}{r_k} \sim G(\overline{x}_k, \overline{t}_k)$. Therefore by the convexity estimate we have that $G(y_k, \tilde{t}_k) \gg G(\overline{x}_k, \overline{t}_k)$. In particular, the curvature derivative estimate holds at the curvature scale $G(y_k, \tilde{t}_k)$. Therefore, the Neck Detection Lemma can be applied at y_k. To summarize, near the point y_k, $M_{\tilde{t}_k}$ looks like a long cylinder, and furthermore $M_{\tilde{t}_k}$ encloses a cone with opening angle $\sim \frac{1}{100}$. This setup contradicts elementary geometry.

□