Curvature Flow of Curves in $\mathbb{R}^n$

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You have a parameterized closed curve γ_t in $\mathbb{R}^n$ and you flow it by normal motion by its curvature vector, given by $\frac{\partial \gamma_t}{\partial t} = k_{\gamma_t}$. There are three major conjectures which are

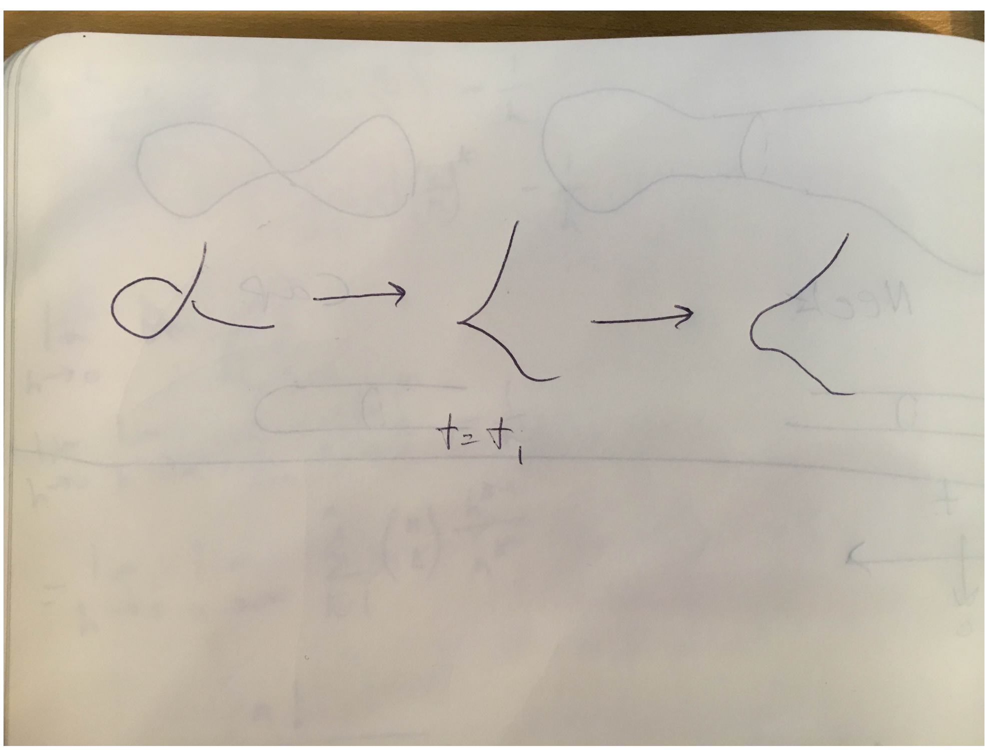
1. No hairs,
2. Spacetime singularity set is finite,
3. A generic flow is smooth (Huisker-Hamilton).

Some results regarding singularities are given by

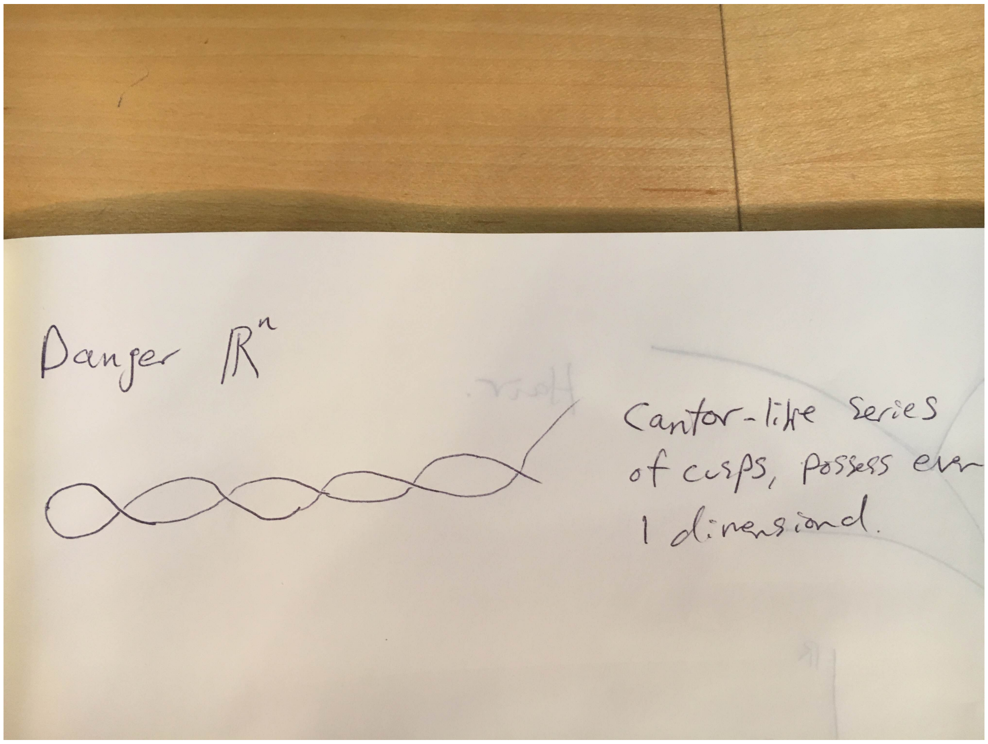
1. In $\mathbb{R}^2$: Gage-Hamilton '84, Grayson '87, Angenent '88, '91, Angenent-Velazquez '95
2. In $\mathbb{R}^n$: Altschuler '91, Altschuler-Grayson '92, Deckelnick '97, '99, Perelman '03, Morgan-Tian '07.

Even if the flow is smooth for some time it eventually becomes singular. You can pass through a simple cusp and there will be no cusp. The cusp at $t = t_1$ can be given by

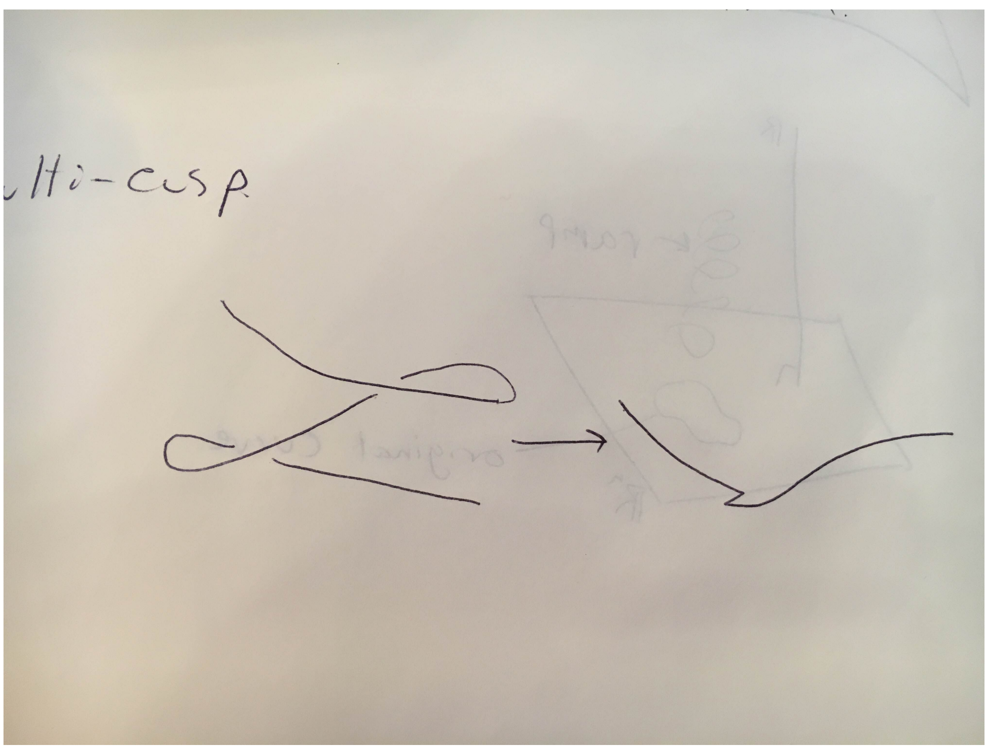
$$(1 + o(1))\,\frac{\pi}{4}\,\frac{|x|}{\log\log\left(\frac{1}{|x|}\right)}.$$



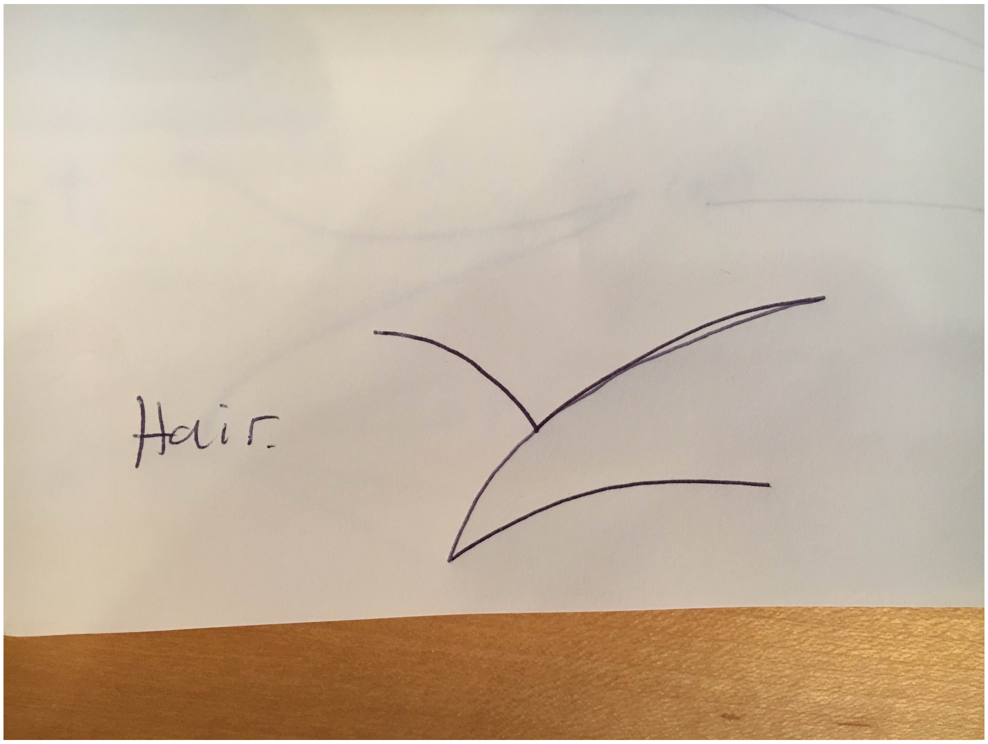
We can prove that the singular set in $\mathbb{R}^n$ is at most one-dimensional.



The next kind of singularity that we might have is a multi-cusp, which has two or more cusps occurring at the same point. It's important to establish whether or not multi-cusps exist or not.



The other kind of danger is a singularity called a *hair*. What if there are only finitely many singular points with C^1 arcs in between them, but such that the curves back-tracks on itself. Grayson was able to rule out these singularities in a special situation in $\mathbb{R}^2$, but it's not clear in $\mathbb{R}^n$.



Theorem 1 (Altschuler-Grayson '92, P '03) *Given a smooth immersion* $\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^3$.

1. *there exists a flow* $(\gamma_t)_{t\in U}$ *defined* C^∞ *on an open dense set of times* $U \subseteq [0,T]$. *Deckelnick: you get an improvement on this, we have that the Hausdorff measure* $\mathcal{H}^{1/2}([0,T) \setminus U) < \infty$,

2. *this flow is unique up to reparametrization in the following sense: if*

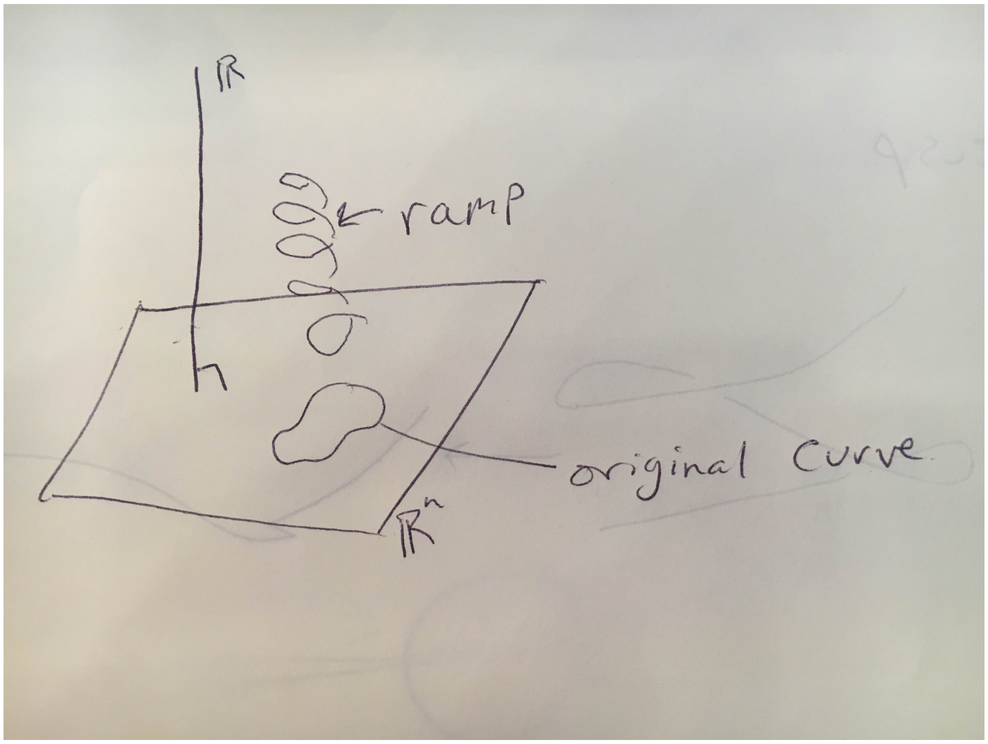
$$d(\gamma_t, \delta_t) = \inf\{area(A) \mid A : \mathbb{S}^1 \times [0,1] \text{ annulus between } \gamma_t, \delta_t\}$$

   then this distance is non-increasing (i.e. is a contraction).

3. *the flow is continuous in annular distance,*

4. *the area of the minimal spanning disk* $\downarrow 0$ *as* $t \uparrow T$ *where* T *is the final time (we would like our function to contract to a point, but there is the possibility of contracting to a hair).*

One method to prove existence is using *ramps*. If you have $\mathbb{R}^n$ you can add one more spatial direction and add a certain slight velocity in the curve in the extra dimension. *Cycles* are like ramps except that we are adding a copy of $\mathbb{R}^2$. If γ_0 is the original curve then we define γ_0^ϵ by



$$\gamma_0^\epsilon(a) \;=\; (\gamma_0(a), \epsilon e^{ia})$$

where $a \in \mathbb{R}$.

Theorem 2 *Let γ_t with $t \in U[0,T]$ be a flow. Then γ_t possesses left and right limits in the uniform topology $C^0(\mathbb{S}^1, \mathbb{R}^n)$ for each $t \in [0,T]$.*

This hinges on the fact $|\gamma_t(a) - \gamma_s(a)|$ is controlled by $|t-s|$ and

$$\int_a^b \int_{\gamma_t} |k|^2 d\mu_t dt.$$

When you have a cusp there is a positive length that lies within the cusp. Suppose you have a sequence of flows, and define $\sigma_i := |k_i|^2 d\mu_{\gamma_t^i} dt \to \sigma$. When you pass through a cusp you have a continuous curve no matter what, you may only come accross discontinuities at a hair.

Definition 1 (Singular Set) *Consider $P_r(x,t) = B_r(x) \times (t-r^2, t+r^2)$. We can decompose $\mathbb{S}^1 \times [0,T] = R \cup H \cup S$, where R consists of th **regular** points (union of smooth immersions), where H is the **past-regular** or **half-regular** points which are regular in $\overline{P_r^-(x,t)}$ for some value of r (up to and including time t they look regular, but cease to behave regularly afterwards), and S represents the **singular** points. **Nonregular** points are those in $H \cup S$, which is a closed set.*

Later on it turns out that H consists of hairs and S consists of cusps.

Theorem 3 *The 1-dimensional parabolic Hausdorff measure of $\gamma_t(H \cup S)$ satisfies*

$$\mathcal{H}_p^1(\gamma_t(H \cup S)) \;\le\; CL(\gamma_0)$$

where $L(\gamma_0)$ is the length of the curve at $t = 0$. There is a subtlety here in that we are taking the image of γ_t. Another way of saying this is that we extend γ_t along to all $t \in [0,T)$ by taking limits from the left.

One of things the above theorem implies is Deckelnick's result. There is however an additional spatial localization here, which can be summarized by an ϵ-regularity lemma where we have that

$$\iint_{P_r^-} |k|^2 \;\leq\; \epsilon r$$

where $|k| \leq \frac{C}{r}$ in the smaller parabolic cylinder $P_{r/2}^-$.