Manifolds with Almost Non-Negative Curvature Operator

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This talk is based on research that was conducted at a workshop last year.

Theorem 1 *For all $n, v_o > 0$ where there exists C, t_o such that (M^n, g) is compact and $Vol(B_1(p)) > v_o$ for all $p \in M$ then we have a lower bound on the curvature operator $Rm \geq -L \geq -1$, where $L > 0$, hence we have an* ***almost non-negative curvature operator****. Then the Ricci flow exists on $[0, t_o]$ and $Rm \geq -CL$. Additionally we find that $|Rm_{g(t)}| \leq \frac{C}{t}$.*

There is an immediate corollary to this theorem.

Corollary 1 *Given n and D, and there exists $\epsilon > 0$ such that if (M^n, g) is compact with diameter $< D$ and volume $\geq v_o$, $Rm \geq -\epsilon$, then M admits metrics with $Rm \geq 0$.*

A few remarks are in order:

- For $n = 3$, a result by Miles Simon
- The analog of the above theorem and corollary holds for almost non-negative complex curvature or PIC I (meaning that $(M, g) \times \mathbb{R}$ has positive isotropic curvature).

If you have a limit of such manifolds as detailed above, you end up with a limit space with a corresponding Ricci flow. There is a problem posed by Petumin, where given a polyhedral complex with angles $\leq 2\pi$ and codimension 2 links, does a Ricci flow come out of it? Another situation to understand when we have an almost non-negative curvature operator. If we restrict our attention to simply connected compact manifolds, then they only examples are torus bundles over symmetric spaces.

Proof of Corollary: Suppose that such an ϵ does not exist. Then we can find a sequence of such manifolds $(M_i^n g_i)$ with $Rm \geq -\frac{1}{i}$. In the limit we have that $(M_i, g_i(t))_{t\in[0,t_0]}$ and obtain a limit manifold $(M_\infty, g_\infty(t))_{t\in[0,t_o]}$ by a result by Hamilton. Then the limit manifold, by the theorem, has non-negative curvature operator, which contradicts our assumption. □

Proof of Theorem: Without loss of generality let $0 < L < L_0 < 1$. Let (M, g) be such an initial manifold, and define a stopping time $t_1 \leq 1$ as the maximal time such that $Vol(B_{g(t)}(1, t)) \geq \frac{v_o}{2}$ and $Rm_{g(t)} \geq -1$. We claim there exists a constant C_1 with $|Rm_{g(t)}| \leq \frac{C_1}{t}$ for all $t \in [0, t_o]$, where C_1 depends only on n and v_o. If we suppose otherwise, then we would find there would exist a sequence (M_i, g_i) of initial manifolds and (p_i, t_i) such that $|Rm_{g(t_i)}(p_i)| \geq \frac{i}{t_i}$. We can then rescale such that $|Rm_{\lambda g(t_i)}(p_i)| = 1$, and thus $\lambda g(t_i)$ will converge to an ancient solution of the Ricci flow with nonnegative curvature operator $Rm \geq 0$ and enclosed volume. Let $-l(p, t)$ denote the smallest eigenvalue of $Rm_{g(t)}(p)$. Then this satisfies an evolution inequality $\frac{\partial l}{\partial t} \leq \Delta l + C\|Rm\| \cdot l$. It turns out that the constant C matters quite a bit. If S is the scalar curvature then we have that C can be taken as $C = S + n^2 l$. If the initial manifold contains an $\mathbb{S}^2$ factor of constant curvature n^2, then the evolution equation becomes simpler:

$$\frac{\partial l}{\partial t} \leq \Delta l + Sl.$$

Now we have the initial condition that $l(p,0) \leq L$, and we consider the equation $\frac{\partial h}{\partial t} = \Delta h + Sh$ with $h(p,0) = L$ and $l(p,t) \leq h$. Now,

$$h(p,t) \;=\; \int_M G(p,t,q,0)d\mu(q)$$

where G is the Green's function and $G(\cdot,\cdot,q,0)$ is a solution to the evolution equation for h. Now $-\Delta_q G(p,t,\cdot,\cdot) = \frac{\partial G}{\partial s}(p,t,\cdot,\cdot)$. The claim now is that when $s < \frac{t}{2}$, $G(p,t,q,s) \leq \frac{C}{t^{n/2}} e^{-d_s^2(p,q)} Ct$, in other words it has a Gaussian bound. □

Theorem 2 *Suppose we have a sequence* (M_i, g_i) *with diameter* $< D$ *and Ricci curvature* $Rm \geq -(n-1)$, *and suppose that* $(\widetilde{M}_i, \widetilde{g}_i)$ *of universal covers* $\to^{GH} \mathbb{R}^n$. *Then for all* i *the Ricci flow exists on* $[0,\infty)$ *and*

$$\lim_{t\to\infty} diam(M_i, g_i(t))^2 \|Rm_{g_i(t)}\| \;\to\; 0.$$