Hermitian Curvature Flows

Gang Tian

Beijing University and Princeton University

Let (M, g, J) be a compact Hermitian manifold of complex dimension n, where J denotes an integrable complex structure and g is compatible with J

$$g(J\cdot, J\cdot) = g(\cdot, \cdot).$$

We denote by $\omega = g(J\cdot, \cdot)$ the Kähler form associated to it.

Is there a geometric method of studying these manifolds?

We say that g is a *Kähler metric* if

$$d\omega = 0.$$

The Ricci flow preserves this Kählerian condition and provides a very important tool for studying Kähler manifolds. In particular, there is a program of classifying Kähler manifolds birationally by Kähler-Ricci flow as Jian Song and I advocated. Many works have been done due to the efforts of a number of mathematicians.

However, the Ricci flow does not preserve Hermitian structures on non-Kähler manifolds.

Let $S = \{S_{k\bar{l}}\}$ be the "Ricci" curvature of the Chern connection of $g = (g_{i\bar{j}})$:

$$S_{k\bar{l}} = -\, g^{i\bar{j}} \left(\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z_i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j} \right).$$

Let $T = \{T_{k\bar{l}i}\}$ be the torsion of g:

$$T_{k\bar{l}p} = \frac{\partial g_{k\bar{l}}}{\partial z_i} - \frac{\partial g_{i\bar{l}}}{\partial z_k}.$$

Note that g is Kähler if and only if $T = 0$.

J. Streets and I introduced a family of new curvature flows on Hermitian metrics $g(t)$:

$$\frac{\partial g}{\partial t} = -S + \hat{Q}(T),$$

where $\hat{Q}(T)$ is a quadratic function of torsion T.

These flows coincide with the Kähler-Ricci flow if the initial metric is Kähler.

We prove: For any given Hermitian metric g_0, the above flow has a unique solution $g(t)$ on $[0, T)$ with $g(0) = g_0$.

Streets and I also developed an analytic theory for the pluri-closed flow analogous to that by Hamilton-Shi for Ricci flow. this includes the evolution equations for curvature and torsion, derivative estimates.

For instance, we proved that if the maximal existence time is $T < \infty$, then

$$\lim_{t \to T} \sup\{|\Omega| + |\nabla T| + |T|\} = \infty.$$

There are two flows which are of special properties:

1. $\hat{Q} = Q$, where $Q_{k\bar{l}} = g^{i\bar{j}} g^{p\bar{q}} T_{k\bar{q}i} \overline{T_{l\bar{p}j}}$.

2. $Q = 0$.

We abbreviate the first flow as PCF and the second as HCF.

Let us discuss PCF. First we recall the pluri-closed condition.

Let (M, g, J) be a Hermitian manifold and ω be its Kähler form. The form g is pluriclosed if

$$\partial\bar{\partial}\,\omega \;=\; 0.$$

As Gauduchon showed in 1977: There exists a unique $u \in C^\infty(M)$ such that $\int_M u dV = 0$ and $\partial\bar{\partial}\left(e^{2u}\omega\right)^{n-1} = 0$.

Pluri-closed metrics always exist on complex surfaces.

If a complex manifold M admits a symplectic structure ω whose (1,1)-part is positive, then it is a pluriclosed manifold.

Streets and I proved that if $\hat{Q} = Q$, then the corresponding Hermitian curvature flow preserves the pluri-closed condition, that is, if g_0 is pluri-closed, so does every $g(t)$ along the flow.

In fact, we proved that in terms of Kähler forms, the flow is equivalent to

$$\frac{\partial \omega}{\partial t} = \partial \partial^*_\omega \omega + \bar{\partial} \bar{\partial}^*_\omega \omega + \sqrt{-1}\, \partial \bar{\partial} \log \det g.$$

Clearly, it implies the pluri-closed condition is preserved.

There is another formulation through the Bismut connection ∇ which is defined via

$$< \nabla_X Y, Z > = < D_X Y, Z > + \frac{1}{2} d\omega(JX, JY, JZ),$$

where D denotes the Levi-Civita connection of ω and J denotes the complex structure.

Let P be the Chern form of this connection, i.e. in complex coordinates,

$$P_{ij} = \Omega^k_{ijk},$$

where Ω denotes the curvature of ∇. Then

$$P = S - dd^* \omega.$$

Hence, the pluri-closed flow is the same as

$$\frac{\partial \omega}{\partial t} = -P^{1,1}.$$

This turns out to be a very useful formulation. For instance, we proved that if the maximal existence time is $T < \infty$, then

$$\int_0^T |P^{1,1}| = \infty.$$

This extends a result of N. Sesum. Also, using this formulation, we can identify our flow with the renormalization group flow twisted with B-fields in theoretic physics.

Let $(M, \omega(t), J)$ be a solution of pluriclosed flow. Define $H = -d\omega(J, J, J)$. Then H is closed since ω is pluriclosed. Let $X = (-Jd^*\omega)^\flat$ and ϕ_t be its integral curve, then we have

- (Streets-Tian) $(\phi_t^*(g(t)), \phi_t^*(H(t)))$ is a solution to the renormalization group flow twisted with B-fields.

We should emphasis that it is a prior unclear why the renormalization group flow twisted with B-fields preserves the pluri-closed condition. One needs to figure out how complex structures evolve along the renormalization group flow.

Applying a result of Oliynyk-Suneeta-Woolgar, we can show that F is monotonic along the pluri-closed flow, where

$$F(g, H, f) := \int_M \left(R - \frac{1}{12}|H|^2 + |\nabla f|^2 \right) e^{-f} dV.$$

In particular, there are no periodic solutions to the pluri-closed flow.

Define

$$\mathcal{H}^{1,1}_{\partial+\bar\partial} = \frac{\{\phi \in \Lambda^{1,1}_{\mathbb{R}} \,|\, \partial\bar\partial\phi = 0\}}{\{\partial\gamma + \bar\partial\bar\gamma \,|\, \gamma \in \Lambda^{0,1}\}}.$$

Furthermore, in analogy with the Kähler cone, let

$$\mathcal{P}_{\partial+\bar\partial} = \{[\phi] \in \mathcal{H}^{1,1}_{\partial+\bar\partial} \,|\, \exists\gamma, \phi + \partial\gamma + \bar\partial\bar\gamma > 0\}.$$

Conjecture: Let $\omega(t)$ be a solution to PCF on M. Let

$$\tau \ = \ \sup\{t > 0 \,|\, [\omega(t)] \in \mathcal{P}_{\partial+\bar\partial}\}.$$

Then the solution exists on $[0, \tau)$.

This cone can be characterized in a nice way on complex surfaces:

Streets-Tian: Let (M^4, J) be a complex, non-Kähler surface, and let $\phi \in \Lambda^{1,1}$ be pluriclosed. Then $\phi \in \mathcal{P}_{\partial + \bar{\partial}}$ if and only if

- $\int_M \phi \wedge \gamma_0 \ > \ 0$
- $\int_D \phi \ > \ 0$ for every effective divisor with negative self intersection.

A generalized Kähler manifold, denoted by (M, g, J_A, J_B), consists of a smooth manifold with two integrable complex structures J_A, J_B and a metric g which is compatible with both, satisfying:

$$d^c_A \omega_A = H = -d^c_B \omega_B, \qquad dd^c_A \omega_A = 0.$$

These equations were discovered by Gates, Hull, and Rocek in 80s in their studying the supersymmetric sigma models and later "rediscovered" by Gualtieri.

• (Streets-Tian) Pluriclosed flow preserves generalized Kähler structure, after coupling with evolution equations for J_A, J_B.

A special case of generalized Kähler geometry occurs when $[J_A, J_B] = 0$. In this case one has a symmetric endomorphism $Q = J_A J_B$ which satisfies $Q^2 = Id$. Thus Q has eigenvalues ± 1 with the corresponding eigenspace decomposition

$$TM = T_+M \oplus T_-M.$$

The commuting generalized Kähler manifolds include products of Riemann surfaces, diagonal Hopf surfaces, Inoue surfaces.

• (J. Streets) Pluriclosed flow preserves commuting generalized Kähler geometry, and moreover reduces to a parabolic flow of a scalar potential function.

The bundles $T_{\pm}^{1,0}$ have determinants $\Lambda^k(T_+^{1,0}), \Lambda^l(T_-^{1,0})$. These in turn have first Chern classes $\rho_{\pm}$.
Consider projections: $\pi_{\pm} : \Lambda^{1,1}(T^*M) \to \Lambda^{1,1}(T_{\pm}^*M)$, put $\psi^{\pm} = \pi_{\pm}\psi$. Then the flow becomes

$$\frac{\partial \omega}{\partial t} = -(\rho_+^+ - \rho_+^- - \rho_-^+ + \rho_-^-).$$

Locally, in coordinates $(z, w) \in U \subset \mathbb{C}^k \times \mathbb{C}^\ell$, we have

$$\omega = \sqrt{-1}(\partial_z \bar{\partial}_z u - \partial_w \bar{\partial}_w u),$$

where u is smooth such that $\sqrt{-1}\partial_z \bar{\partial}_z u > 0$ and $\sqrt{-1}\partial_w \bar{\partial}_w u < 0$.

So the pluri-closed flow in this case is a fully nonlinear, nonconvex scalar PDE.

(J. Streets): If $(M, g_0, J_{\pm})$ is a non-Kähler, generalized Kähler surface satisfying $[J_+, J_-] = 0$, then for any initial g_0, the pluri-closed flow has a global solution. Also if M is a surfaces of general type, the normalized pluriclosed flow exists for all time and converges to the unique Kähler-Einstein metric on (M, J_+).

This is the first result on the global existence of the pluri-closed flow and shows that its restriction to generalized Kähler metrics resembles the Kähler-Ricci flow.

To prove the result, one needs to prove a priori bound for the potential function u, an upper bound for the metric which can be derived from an evolution equation for $\partial_+\partial_- u$ and a $C^{2,\alpha}$-estimate for u.

For the Kähler-Ricci flow, the corresponding $C^{2,\alpha}$ is obtained by applying the Evans-Krylov theory.

In our case, the equation is nonconvex, so the Evans-Krylov theory does not apply. Streets-Warren extended the Evans-Krylov theory to certain nonconvex equations which is sufficient for the above case.

A result of Streets-Warren: If $u : \mathbb{C}^k \times \mathbb{C}^l \to \mathbb{R}$ be a solution of

$$\frac{\det u_{z\bar{z}}}{\det -u_{w\bar{w}}} = 1$$

on $B_2(0)$ satisfying: $\sqrt{-1}\partial_z\bar{\partial}_z u >$**0** and $\sqrt{-1}\partial_w\bar{\partial}_w u < 0$. Then there exists $C, \gamma > 0$ such that

$$||u||_{C^{2,\gamma}(B_1)} \leq C||u||_{C^{1,1}(B_2)}.$$

The other extremal case is when the 2-form σ due to $[J_A, J_B]$ is non-degenerate. There are examples of such generalized Kähler structures.

Recently, J. Streets proved that given a non-degenerate generalized Kähler surface, the generalized Kähler-Ricci flow with this initial data exists for all time and converges to a weakly hyperKähler structure.

Next we consider

$$\frac{\partial g}{\partial t} = -S.$$

In the following, I will briefly discuss some recent results of Yury Ustinovskiy on this flow.

Given any Hermitain manifold (M, ω, J), we denote by Ω the curvature of the Chern connection. We say (M, ω, J) is Griffiths non-negative if for any $\xi, \eta \in T^{1,0}M$, we have

$$\Omega(\xi, \bar{\xi}, \eta, \bar{\eta}) \geq 0.$$

Ustinovskiy proved that if the initial metric has Griffiths non-negative, so does $g(t)$ along the Chern flow. Moreover, if the Chern the Chern curvature at $t = 0$ is Griffiths-positive at some point $x \in M$, then for any $t > 0$, the Chern curvature is Griffths-positive everywhere.

Ustinovskiy also proved a strong Maximum principle analogous to that of Hamilton for Ricci flow:

Let $g(t)$ ($t \in [0, \tau)$) be a solution to Hermitian curvature flow. Assume that the Chern curvature at $t = 0$ is Griffiths non-negative. Then for any $t > 0$ the set

$$Z = \{(\xi, \eta) \,|\, \xi, \eta \in T^{1,0}M, \;\; \Omega(\xi, \bar{\xi}, \eta, \bar{\eta}) = 0\}$$

is invariant under torsion-twisted parallel transport

$$\nabla_{\gamma'}\xi = T(\gamma', \xi), \quad \nabla_{\gamma'}\eta = 0.$$

He also proved a better theorem on maximal existence time:

Let $g(t)$ ($t \in [0, \tau)$) be a solution to HCF on a compact complex Hermitian manifold. Assume that $[0, \tau)$ is the maximal time interval on which solution to HCF exists. Then

$$\overline{\lim_{t \to \tau}} ||\Omega||_{C^0} = \infty.$$

In the proof, Ustinovskiy found very nice formulation for curvature and torsion evolution equations: In moving unitary frame:

$$\frac{D\Omega}{dt} = \Delta^T \Omega + \mathcal{T}(\Omega) + \mathcal{F}(\Omega),$$

where $\mathcal{T}(\Omega)$ is linear in Ω with coefficients given by ∇T and $\mathcal{F}(\Omega)$ is the same as that in Hamilton's Ricci flow.

$$\frac{DT}{dt} = \Delta T + \mathcal{Q}(T),$$

where $\mathcal{Q}(T)$ is linear in T with coefficients given by Ω.

Here are definition of $\mathcal{T}$ and $\mathcal{Q}$ and $\mathcal{F}$:

$$\mathcal{T}(\Omega)(\xi,\bar{\xi},\eta,\bar{\eta}) \;=\; -2\mathrm{Re}\sum\Omega(\nabla_{\bar{e}_i}T(\xi,e_i),\bar{\xi},\eta,\bar{\eta}).$$

$$\mathcal{Q}(T)(\xi,\eta) \;=\; \sum\Omega(T(e_i,\eta),\bar{e}_i)\xi-\Omega(T(e_i,\xi),\bar{e}_i)\eta.$$

Also Δ^T is the Laplacian of ∇^T which is defined by

$$\nabla^T_e\xi=\nabla_e\xi+T(\xi,e).$$