Hypersurfaces of Low Entropy

Lu Wang

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The geometry and topology of hypersurfaces of low entropy. First we will introduce the Gaussian surface area and entropy. Next we will state the Gap Theorem/Conjecture on this entropy, and then give main results in this direction (joint work with J. Bernstein). If time permits we will talk about some of the key aspects of these results, namely:

- the asymptotic behavior of shrinkers, and
- the topology of shrinkers of low entropy.

Denote by $\Sigma^n \subseteq \mathbb{R}^{n+1}$ a Gaussian surface area of Σ is defined to be

$$F[\Sigma] = (4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x|^2}{4}}.$$

In particular we have that $F[\mathbb{R}^n \times \{0\}] = 1$. The E-L equation is given by

$$H = \frac{1}{2}\langle x, U\rangle \tag{1}$$

We call a solution of equation 1 a **self shrinker**. Denote by $\mathbb{S}^n(\sqrt{2n})$ the sphere in $\mathbb{R}^{n+1}$ of radius $\sqrt{2n}$. Common surfaces we shall consider are $\mathbb{R}^n \times \{0\}$, $\mathbb{S}^n(\sqrt{2n})$, and $\mathbb{S}^k(\sqrt{2n}) \times \mathbb{R}^{n-k}$. The self-adjoint operator associated to the 2nd variations is given by

$$L = \underbrace{\Delta - \frac{1}{2}\langle x, \nabla\rangle}_{\text{Orstein-Uhlenbeck Operator}} + \left(|A|^2 + \frac{1}{2}\right).$$

If Σ is a self-shrinker then $L\langle y, U\rangle = \frac{1}{2}\langle y, U\rangle$ for $y \in \mathbb{R}^{n+1}$ and $L\langle x, U\rangle = \langle x, U\rangle$.

Corollary 1 (Colding-Minicozzi) *There are no stable self-shrinkers with polynomial volume growth.*

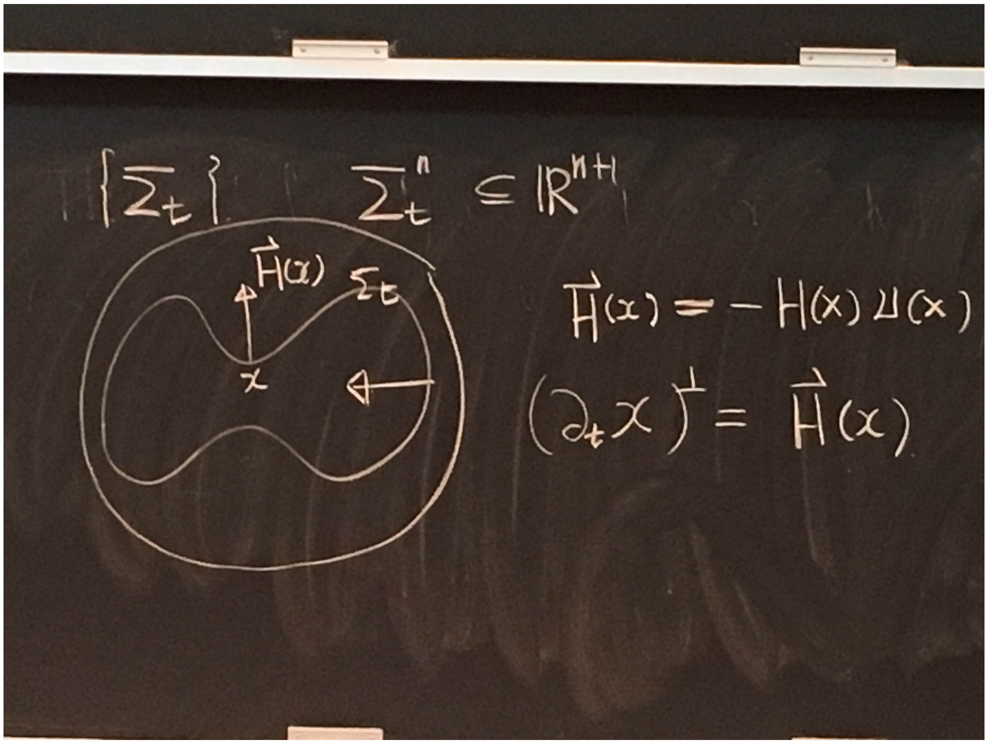
Colding-Minicozzi introduce the entropy of the hypersurface. We denote by λ the operator

$$\lambda[\Sigma] = \sup_{\substack{y \in \mathbb{R}^{n+1} \\ \rho > 0}} F[\rho\Sigma + y]$$

Theorem 1 (Colding-Minicozzi '09) *The only entropy-stable self-shrinkers with polynomial volume growth are*

1. $\mathbb{S}^n(\sqrt{2n})$,
2. $\mathbb{R}^n \times \{0\}$, *or*
3. $\mathbb{S}^k(\sqrt{2n}) \times \mathbb{R}^{n-k}$.

Work by Ketover-Zhou compares minimal hypersurfaces in $\mathbb{S}^{n+1}$ using min-max theory in Gaussian weighted space. Ketover discovered new examples of self-shrinkers in $\mathbb{R}^3$ with discrete symetry and low genus. One motivation to look at the entropy of hypersurfaces has to do with mean curvature flow. Recall that we can define a one curve family $\{\Sigma_t\}$, $\Sigma_t^n \subseteq \mathbb{R}^{n+1}$ of hypersurfaces and consider their mean curvature vector $\vec{H}(x) = -H(x)u(x)$ where $\vec{H}(x) = (\partial_t x)^\perp$. We would like to think about the space-time track of this flow.



Consider parameterizations $X = (x,t)$ and $Y = (y,s)$ where the second coordinate is along the time direction. We define a distance by

$$d(X,Y) \;=\; |x-y| + \sqrt{|t-s|}.$$

An important tool in this analysis is Huisken's monotonicity formula given by

$$\begin{aligned} \frac{d}{dt}\int_{\Sigma_t} \Phi_{(x_0,t_0)} &= -\int_{\Sigma_t} \left| H - \frac{\langle x - x_0, U\rangle}{2(t_0 - t)} \right|^2 \Phi_{(x_0,t_0)} \\ \Phi_{(x_0,t_0)}(x,t) &= (4\pi(t_0-t))^{-\frac{n}{2}}\, e^{-\frac{|x-x_0|^2}{4(t_0-t)}}. \end{aligned}$$

Corollary 2 *1. The singularity models of mean curvature flow are self-shrinkers.*

2. λ is monontone decreasing under mean curvature flow.

3. If Σ is a self-shrinker then $\lambda(\Sigma) = F(\Sigma)$.

A result by Stone yields the inequality

$$2 \;>\; \lambda(\mathbb{S}^1) \;>\; \frac{3}{2} \;>\; \lambda(\mathbb{S}^2) \;>\; \ldots \;>\; \lambda(\mathbb{S}^n) \;>\; \ldots \;\to\; \sqrt{2}$$

as well as the fact that $\lambda(\mathbb{R}^n \times \{0\}) = 1$.

Theorem 2 (Brake, Huisken, White) *There exists $\delta_0(n) > 0$ such that if Σ^n is a non-flat shrinker in $\mathbb{R}^{n+1}$ then $\lambda(\Sigma) > 1 + \delta_0(n)$.*

Here we have $H = \frac{1}{2}\langle x, U\rangle$ on Σ and $\Sigma_t = \sqrt{-t}\Sigma$ where $\{\Sigma_t\}_{t<0}$ is a mean curvature flow.

Conjecture 1 (Colding-Ilmanen-Minicozzi-White, '12) *For $2 \leq n \leq 6$ there exists $\delta_1(n) > 0$ such that if Σ is a non-flat and non-rounded self-shrinker in $\mathbb{R}^{n+1}$, then the entropy $\lambda(\Sigma) > \lambda(\mathbb{S}^n) + \delta_1$.*

Conjecture 2 (C-I-M-W, '12) *For $2 \leq n \leq 6$ if Σ is a closed hypersurface in $\mathbb{R}^{n+1}$ then $\lambda(\Sigma) \geq \lambda(\mathbb{S}^n)$, with equality holding if and only if $\Sigma = \rho\mathbb{S}^n + y$ for some $\rho > 0$ and $y \in \mathbb{R}^{n+1}$ (in other words that the hypersurface is round).*

A few remarks about the conjectures:

1. For $n = 1$ there is nothing to say by a result of Grayson and Gage-Hamilton.
2. We see from $H = \frac{1}{2}\langle x, U\rangle$ the hypersurface will contain minimal cones. The entropy of a Simons cone approaches the limit $\sqrt{2}$ which is the same as the sphere.
3. It happens that the first conjecture is true assuming the self-shrinker is closed (meaning compact and no boundary). This was shown by C-I-M-W.

Theorem 3 (Bernstein-Wang, '14) *The second conjecture is true.*

Remark: Interestingly the proof used to prove the second conjecture does not actually imply that the first conjecture is true.

Theorem 4 (Bernstein-Wang, '15) *For $n \in \{2, 3\}$, if a closed hypersurface Σ satisfies that $\lambda(\Sigma) \leq \lambda(\mathbb{S}^{n-1} \times \mathbb{R})$ then Σ is diffeomorphic to $\mathbb{S}^n$.*

This last theorem implies that there is a topological stability to this entropy. One of the important elements of the proof is the topology of shrinkers from the law of entropy.

Theorem 5 (Bernstein-Wang, '15) *If $\Sigma^n \subseteq \mathbb{R}^{n+1}$ is a shrinker asymptotic to a regular cone and $\lambda(\Sigma) \leq \lambda(\mathbb{S}^{n-1} \times \mathbb{R})$ then the link of the asymptotic cone is connected and separates $\mathbb{S}^n(1)$ into two components that are both diffeomorphic to Σ.*

A refinement of this last theorem gives us that for $n \in \{2, 3\}$ we have that Σ is diffeomorphic to $\mathbb{R}^n$. Another element of the theorem is to study the asymptotics of self-shrinkers.