Ricci flow through singularities

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Short-time existence (Hamilton). If h is a smooth Riemannian metric on a compact manifold M, there is a unique family of metrics $\{g(t)\}_{t\in[0,T)}$ with $g(0) = h$ satisfying the Ricci flow equation

$$\frac{\partial g}{\partial t} = -2\ \mathrm{Ric}(g)\,,$$

defined on a maximal time interval $[0, T)$.

If $T < \infty$ then the curvature blows up as $t \to T$.

Examples.

- If $g(0)$ is the standard metric on the n-sphere S^n, then

$$g(t) = (1 - 2(n-1)t)g(0).$$

Here the curvature blows up as $t \to \frac{1}{2(n-1)}$.

- If $g(0)$ is a "barbell" metric on S^3, then the Ricci flow $\{g(t)\}_{t \in [0,T)}$ will develop a neck pinch singularity.

Here only part of the metric goes singular, while elsewhere it has a smooth limit as $t \to T$.

Q: Can one continue the flow past the first singular time?

Ricci flow with surgery:

- (Hamilton, Chen-Zhu) 4*d* Ricci flows with positive isotropic curvature.
- (Hamilton, Perelman) 3*d* Ricci flow.

Applications of Ricci flow with surgery:

- Thurston's Geometrization Conjecture.
- Classification of 4-manifolds with positive isotropic curvature.

Note. Ricci flow with surgery is not a canonical construction, because it involves a number of choices.

Perelman: "It is likely that by passing to the limit in this construction one would get a canonically defined Ricci flow through singularities, but at the moment I don't have a proof of that."

Q: How can one formalize Perelman's assertion?

Q: What is a Ricci flow through singularities?

Q: What notion of convergence should one use?

Weak/generalized solutions to geometric PDE's:

- Minimal submanifolds.
- Harmonic maps.
- Einstein metrics.
- Harmonic map heat flow.
- Mean curvature flow.

Agenda: Define generalized solutions and study their structure — existence, uniqueness, partial regularity, compactness, etc.

Mean curvature flow

There is a strong analogy between mean curvature flow and Ricci flow, especially between mean curvature flow of surfaces in $\mathbf{R}^3$, and Ricci flow in dimension 3.

There are several notions of generalized solutions to the mean curvature flow equation:

- (Enhanced) Brakke flows.
- Level set flow.
- Flow with surgery.

In the case of mean curvature flow with a smooth initial condition with positive mean curvature, fundamental work of White and Huisken-Sinestrari gives deep understanding of mean curvature flow.

The theory of generalized solutions is very satisfactory in this case. In particular:

- All notions of generalized solution agree in the natural sense, and one has existence of a unique solution.
- The analog of Perelman's assertion is true.

Related work

- Feldman-Ilmanen-Knopf (2003). Kahler-Ricci flow through an isolated singularity.
- Angenent-Knopf (2007), Angenent-Caputo-Knopf (2012). Existence of Ricci flow (with precise asymptotic control) starting from a singular initial condition corresponding to a generic neckpinch.
- Song-Tian (2009), Eyssidieux-Guedj-Zeriahi (2014). Kahler-Ricci flow through singularities.

To formulate the first theorem, I will recall some facts about Ricci flow with surgery.

Disclaimer. Due to time constraints and the nature of Ricci flow with surgery, details have been suppressed.

Def. A **(primitive) Ricci flow with surgery** is given by:

- Ricci flows

$$\{(M_k \times [t_k^-, t_k^+), g_k(\cdot))\}_{1 \le k \le N},$$

  where $t_k^+ = t_{k+1}^-$ for all $1 \le k < N$, and the flow g_k goes singular at t_k^+ for each $k < N$.
- Open subsets $\Omega_k \subset M_k$ where the metric has a smooth limit $\bar{g}_k$ as $t \to t_k^+$, for $k < N$.
- Compact 3-dimensional submanifolds with boundary

$$X_k^+ \subset \Omega_k$$

  which survive the "surgery".
- Isometric embeddings

$$\psi_k : X_k^+ \longrightarrow \psi_k(X_k^+) = X_{k+1}^- \subset M_{k+1}$$

Perelman constructs a Ricci flow with surgery starting at any initial Riemannian 3-manifold $(M, g(0))$, that satisfies several additional conditions:

- **(Controlled surgery)** Cutting is performed along necks of a specified scale and quality, and the surgery caps are well approximated by standard caps.
- **(Hamilton-Ivey pinching)** This implies that the full curvature tensor Rm is controlled by the scalar curvature R, and that when R is large the curvature is "almost nonnegative":
$$\mathrm{Rm} \gtrsim -\frac{R}{\log R}\,.$$
- **(Canonical neighborhoods)** At any point with large scalar curvature R, the flow is well-approximated by a model solution ("κ-solution") or by a standard postsurgery solution.

Def. A **Ricci flow spacetime** is a tuple $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ where:

- $\mathcal{M}$ is a smooth manifold-with-boundary.
- $\mathfrak{t}$ is the **time function** – a submersion

$$\mathfrak{t} : \mathcal{M} \to [0, \infty)$$

  such that the boundary of $\mathcal{M}$ is precisely the time-zero slice: $\partial\mathcal{M} = \mathfrak{t}^{-1}(0)$.
- $\partial_{\mathfrak{t}}$ is the **time vector field**, satisfying $\partial_{\mathfrak{t}}\mathfrak{t} \equiv 1$.
- g is a smooth Riemannian metric on the spatial subbundle

$$\ker(d\mathfrak{t}) \subset T\mathcal{M}\,,$$

- g defines a Ricci flow:

$$\mathcal{L}_{\partial_{\mathfrak{t}}}\, g = -2\,\mathrm{Ric}(g)\,.$$

For $0 \leq a < b$ we write

$$\mathcal{M}_a = \mathfrak{t}^{-1}(a), \quad \mathcal{M}_{[a,b]} = \mathfrak{t}^{-1}([a,b])$$

$$\mathcal{M}_{\leq a} = \mathfrak{t}^{-1}([0,a])\,.$$

For every Ricci flow with surgery in the sense of Perelman, there is an associated Ricci flow spacetime $\mathcal{M}$ obtained by gluing together the spacetimes from the time intervals. Henceforth we will conflate this spacetime with the Ricci flow with surgery.

Theorem. Let $\{\mathcal{M}^j\}_{j=1}^{\infty}$ be a sequence of three-dimensional Ricci flows with surgery (in the sense of Perelman) where:

- The initial conditions $\{\mathcal{M}_0^j\}$ are isometric to a fixed compact Riemannian 3-manifold.
- If $\delta_j : [0, \infty) \to (0, \infty)$ denotes the Perelman surgery parameter for $\mathcal{M}^j$ then $\lim_{j\to\infty} \delta_j(0) = 0$.

Then, after passing to a subsequence, $\{\mathcal{M}^j\}$ converges to a Ricci flow spacetime $(\mathcal{M}^\infty, \mathfrak{t}_\infty, \partial_{t_\infty}, g_\infty)$, in a sense described below.

Informally, this means that up to diffeomorphisms defined on larger and larger subsets, the flows are closer and closer.

There is a sequence of diffeomorphisms

$$\{\Phi^j : \mathcal{M}^j \supset U_j \to V_j \subset \mathcal{M}^\infty\},$$

between open sets such that:

- For all $\bar{t}$, $\overline{R} \in [0, \infty)$, and large j, we have:

$$U_j \supset \mathcal{M}^j_{\leq \bar{t}} \cap \{R \leq \overline{R}\},$$
$$V_j \supset \mathcal{M}^\infty_{\leq \bar{t}} \cap \{R \leq \overline{R}\}.$$

- Φ^j is time preserving.
- The sequences $\{\Phi^j_* \partial_{t_j}\}$, $\{\Phi^j_* g_j\}$ converge smoothly on compact subsets of $\mathcal{M}^\infty$ to ∂_{t_∞} and g_∞, respectively.

Furthermore, $\mathcal{M}^\infty$ inherits geometric features of the Ricci flows with surgery:

- The scalar curvature function $R : \mathcal{M}^\infty \to \mathbf{R}$ is bounded below.
- R is proper on $\mathcal{M}^\infty_{\leq T}$ for all $T \geq 0$.
- $\mathcal{M}^\infty$ satisfies the Hamilton-Ivey pinching condition.
- $\mathcal{M}^\infty$ satisfies a strengthened version of the canonical neighborhood assumption: points with large R are modelled on κ-solutions.

This theorem gives a partial answer to Perelman's question, by formalizing the notion of limit and convergence, and proving subsequential convergence.

Remark. A similar theorem holds for $4d$ Ricci flows with surgery, assuming positive isotropic curvature.

Motivated by the theorem, we make the following definition:

Def. A Ricci flow spacetime $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is a **singular Ricci flow** if it is 4-dimensional, the initial time slice $\mathcal{M}_0$ is a compact Riemannian manifold and:

- The scalar curvature function $R : \mathcal{M}^\infty \to \mathbf{R}$ is bounded below.
- R is proper on $\mathcal{M}^\infty_{\leq T}$ for all $T \geq 0$.
- $\mathcal{M}$ satisfies the Hamilton-Ivey pinching condition.
- $\mathcal{M}$ satisfies the canonical neighborhood assumption.

Remark. It suffices to assume the latter two conditions hold outside of a compact subset of $\mathcal{M}_{\leq T}$, for every T.

Corollary. If N is a compact Riemannian 3-manifold, then there is a singular Ricci flow $\mathcal{M}$ with initial condition isometric to N.

Theorem. (Compactness) If $\{\mathcal{M}^j\}$ is a sequence of singular Ricci flows, and the initial conditions $\mathcal{M}^j_0$ lie in a compact set of metrics in the smooth topology, then a subsequence converges to a limiting singular Ricci flow $\mathcal{M}^\infty$, as in the statement of the previous convergence theorem.

Conjecture. If two singular Ricci flows have the same initial condition, then they are the same, up to a diffeomorphism that preserves the time functions, the time vector fields, and the metrics.

This conjecture together with the convergence theorem implies an affirmative answer to Perelman's question.

Remark. One could potentially apply Ricci flow to families of Riemannian metrics. One would need uniqueness in order to get continuous dependence of the flow on the initial condition.

Ingredients in the proof of the convergence theorem

- **Spacetime metric.** If $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is a Ricci flow spacetime, there is a unique Riemannian metric $g_{\mathcal{M}}$ on $\mathcal{M}$ whose restriction to $\ker(d\mathfrak{t})$ is g, and such that $\partial_{\mathfrak{t}}$ is a unit vector field orthogonal to $\ker(d\mathfrak{t})$.

- **Locally controlled geometry.** The canonical neighborhood assumption implies that asymptotically, the geometry of the Riemannian manifold $(\mathcal{M}, g_{\mathcal{M}})$ is controlled locally, as a function of the scalar curvature function $R : \mathcal{M} \to \mathbf{R}$.

- **Compactness for spacetimes.** The collection of pointed spacetimes with locally controlled geometry is compact.

- **Connectedness to $\mathcal{M}_0$ with controlled R.** Any point x in $\mathcal{M}_t^j$ can be joined to the initial time slice by a time preserving curve with controlled length and scalar curvature.

Def. If $\mathcal{M}$ is a Ricci flow spacetime, a path

$$\gamma : [a, b] \to \mathcal{M}$$

is **time-preserving** if $\mathfrak{t}(\gamma(t)) = t$ for all $t \in [a, b]$.

In the remainder of the talk, I will discuss some additional results about singular Ricci flows, that are not a priori, limits of Ricci flows with surgery.

These are of interest both because they say something about the limiting behavior of Ricci flow with surgery, and because they show that singular Ricci flows are good objects.

In what follows, $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ will be a fixed singular Ricci flow.

Theorem. (Behavior of volume)

- For every t, the time slice $\mathcal{M}_t$ has finite volume.
- $t \mapsto \mathrm{vol}(\mathcal{M}_t)$ is a locally Holder continuous function with a locally bounded upper right derivative.
- In the convergence theorem, the sequence of volume functions

$$\{V_j : [0,\infty) \to [0,\infty]\}_j\,, \quad V_j(t) = \mathrm{vol}(\mathcal{M}^j_t)$$

converges uniformly on compact time intevals to the volume function

$$V_\infty(t) = \mathrm{vol}(\mathcal{M}^\infty_t)\,.$$

- R is integrable on $\mathcal{M}_{\leq T}$ for all T.
- For $t_1 < t_2$, the usual formula holds:

$$\mathrm{vol}(\mathcal{M}_{t_2}) - \mathrm{vol}(\mathcal{M}_{t_1}) = - \int_{\mathcal{M}_{[t_1, t_2]}} R \, dvol \, .$$

- For every $a < 1$:

$$\int_{\mathcal{M}_t} R^a \, dvol < C = C(\mathcal{M}_0, t, a) \, .$$

  In particular

$$\mathrm{vol}(\mathcal{M}_t \cap \{R \geq \underline{R}\}) \to 0$$

  as $\underline{R} \to \infty$.

- **Structure of the thin part.** For every t, the part of $\mathcal{M}_t$ with large R has standard geometry and topology. It is contained in a disjoint union of connected components $\{C_i\}$ where each C_i is diffeomorphic to $\mathbf{R}^3$, a spherical space form, or an isometric quotient of $S^2 \times \mathbf{R}$.

- **Ends.** Each connected component of $\mathcal{M}_t$ has only finitely many ends, and taking the completion adds at most one point for every end.

- **Components persist backward in time.** If $\gamma_1, \gamma_2 : [a, b] \to \mathcal{M}$ are time preserving curves, and $\gamma_1(b)$, $\gamma_2(b)$ lie in the same connected component of $\mathcal{M}_b$, then $\gamma_1(t)$, $\gamma_2(t)$ lie in the same component of $\mathcal{M}_t$ for all $t \in [a, b]$.

Causal structure

Def. If $\mathcal{M}$ is a Ricci flow spacetime, the **worldline** of $x \in \mathcal{M}$ is the maximal time-preserving integral curve of ∂_{t} passing through x. A **bad worldline** is a worldline $\gamma : I \to \mathcal{M}$ that doesn't start at time zero.

Theorem. Let $\mathcal{M}$ be a singular Ricci flow, $t \geq 0$. For any connected component C of $\mathcal{M}_t$, there are only finitely many points in C with bad worldlines. In particular, there are only countably many bad worldlines in $\mathcal{M}$.

Example. If one has a generic neckpinch singularity at $t = T$, then the theorem says that only finitely many worldlines (in this case two) can emerge from the singularity.

The main part of the proof of the theorem is devoted to showing that if $\gamma : (t_0, T) \to \mathcal{M}$ is a bad worldline, then for t close to t_0, $\gamma(t)$ is trapped in a "fingertip" which goes singular as $t \searrow t_0$.

This is based on a new dynamical stability result for cylinders.

The fact that the union of the bad worldlines has measure zero plays a key role in the proof of the properties of the volume.