The Ricci Flow on the Sphere with Marked Points

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May 4, 2016

The talk today will be about Ricci flow on Riemann surfaces.

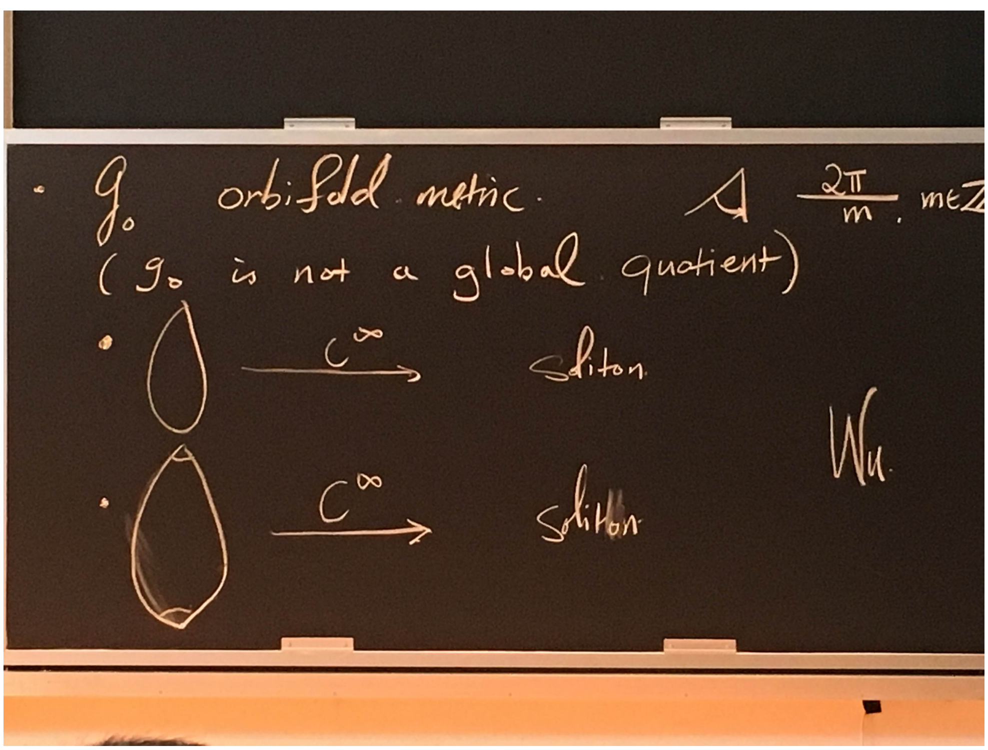
1 Background

If we look at the standard Ricci flow $\frac{\partial g}{\partial t} = -2Ric(g)$ with initial condition $g|_{t=0} = g_0$ on $\mathbb{S}^2$. There is a result that

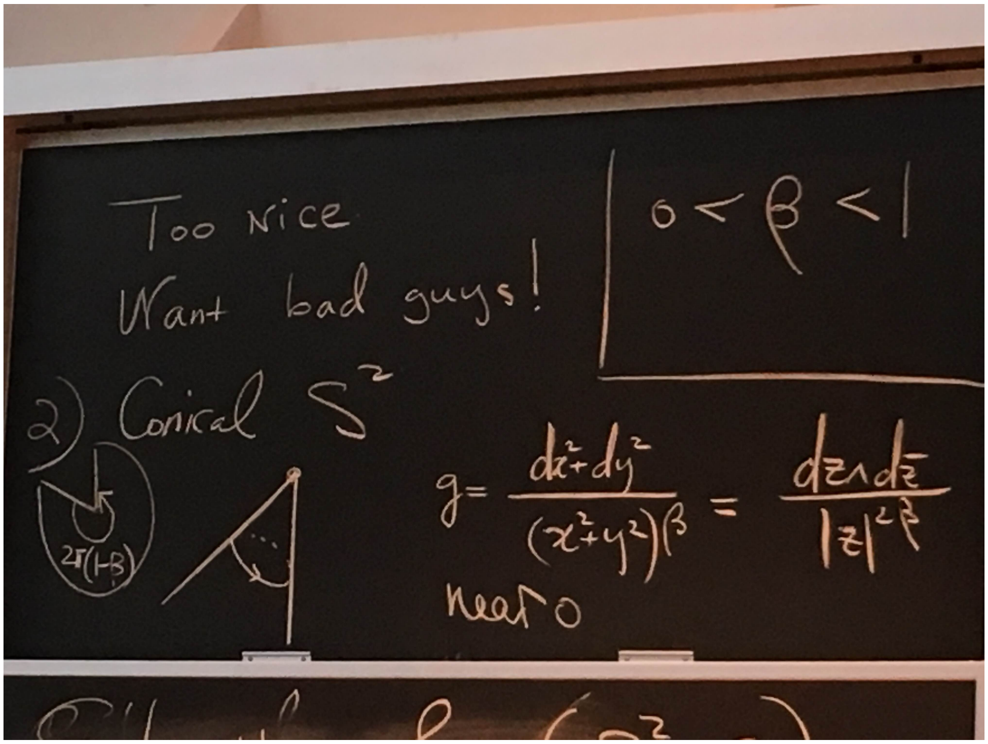
$$\frac{g(t)}{\int_{\mathbb{S}^2} dg(t)} \overbrace{\longrightarrow}^{C^\infty} g_{\mathbb{S}^2}.$$

We can pick g_0 to be an orbifold metric, it's important to note that g_0 may not be a global quotient (above proven by Hamilton, Chow, Struwe). If not then we obtain two cases:

1. We can obtain a teardrop converging to a soliton.
2. The other case we have two orbifold singularities and this converges to a nontrivial soliton.



In both cases we have C^∞ convergence (result by Wu). This construction is in some ways "too nice." We would like to consider cases where we have a generalization of orbifold singularities, given by the conical $\mathbb{S}^2$. Consider the metric



$$g \;=\; \frac{dx^2+dy^2}{(x^2+y^2)^\beta} \;=\; \frac{dz\wedge d\overline{z}}{|z|^{2\beta}}$$

near 0, where $0<\beta<1$. We can find distinct points of $\mathbb{S}^2$ to make singular, and we can define a **pair** $(\mathbb{S}^2,\beta)$ where $\beta = \beta_1[p_1] + \ldots + \beta_k[p_k]$ where each $\beta_i \in (0,1)$ (they are weights). Our cones are given by the conical angles $2\pi(1-\beta_i)$. We normally have a conformal/complex structure on $\mathbb{S}^2$, but we arrive at the following question: when does the pair $(\mathbb{S}^2,\beta)$ with conical singularities admit positively constant curvature metrics? We want the constraint that it's positively curved, which constrains the number of singularities we can have. Let's consider now the Euler number of $(\mathbb{S}^2,\beta)$. A straightforward computation shows that

$$\chi(\mathbb{S}^2,\beta) \;=\; \chi(\mathbb{S}^2) - \sum_{j=1}^{k}\beta_j \;=\; 2-\sum_{j=1}^{k}\beta_j \;=\; \int_{\mathbb{S}^2\setminus\beta} R(g)dg \;>\; 0$$

where the last part of the equation is by the Gauss-Bonnet theorem. Curvature in a sense has δ-functions at each singularity. To recall, the main assumptions we are making here is that $0<\beta_i<1$ for all $i\in\{1,\ldots,k\}$ and that

$$\sum_{j=1}^{k}\beta_j \;<\; 2.$$

Definition 1 *Let $k\geq 3$ and consider a pair $(\mathbb{S}^2,\beta)$ satisfying the above conditions, and we order the β_i by $0<\beta_1\leq\beta_2\leq\ldots\leq\beta_k=\beta_{\max}$. We say that the pair is **stable** if $\beta_k<\sum_{j=1}^{k-1}\beta_j$. We say that the pair is **semi-stable** if $\beta_k=\sum_{j=1}^{k-1}\beta_j$, and is **unstable** if $\beta_k>\sum_{j=1}^{k-1}\beta_j$. We refer to this as the k-stability of the pair.*

Theorem 1 (Troyanov, Luo-Tian) *There exists a unique constant curvature metric if and only if the pair* $(\mathbb{S}^2, \beta)$ *is stable. Furthermore, if the pair* $(\mathbb{S}^2, \beta)$ *is not stable, then there is no Einstein metric and there is no soliton.*

2 Main Results

Let's look at the canonical Ricci flow but add the additional terms and conditions

$$\frac{\partial g}{\partial t} = -Ric(g) + \left(2 - \sum_{j=1}^{k} \beta_j\right) g$$

where $g|_{t=0} = g_0$ and $\int_{\mathbb{S}^2} dg_0 = 2$.

Theorem 2 (Phong-Song-Sturm-Wang) *Consider the pair* $(\mathbb{S}^2, \beta)$.

1. *If the pair is stable then* $g(t) \to g_\infty$ *where the convergence is* C^∞_{loc} *and* g_∞ *is an Einstein metric (result by Mazzeo-Rubenstein-Sesum).*
2. *If the pair is semi-stable then* $(\mathbb{S}^2, \beta, g(t)) \to (\mathbb{S}^2, \beta_\infty, g_\infty)$ *where convergence is* d_{GH}, $\beta_\infty = \beta_k[p_\infty] + \beta_k[q_\infty]$ *so the limiting pair has only two singularities, where we have*

$$Ric(g_\infty) = \left(2 - \sum_{j=1}^{k} \beta_j\right) g_\infty$$

   and $p_k \to \infty$ *and* $p_1, p_2, \ldots, p_{k-1} \to q_\infty$ *(the largest singularity goes to one limit where alll other singularities go to a separate limit).*

3. *If the pair is unstable then* $(\mathbb{S}^2, \beta, g(t)) \to (\mathbb{S}^2, \beta_\infty, g_\infty)$ *where convergence is* d_{GH} *and*

$$\beta_\infty = \beta_k[p_\infty] + \left(\sum_{j=1}^{k-1} \beta_j\right)[q_\infty]$$

   and

$$Ric(g_\infty) = \left(2 - \sum_{j=1}^{k} \beta_j\right) g_\infty + L_v g_\infty$$

   and $p_k \to p_\infty$ *while* $p_1, \ldots, p_{k-1} \to q_\infty$.

It's important to note that R is uniformly bounded, but we cannot directly apply Hamilton's compactness theorem. Additionally we know this is a type I solution, but we need a double blow-up because we don't know the singular behavior of the merging points. To do this consider p_1 and p_2, and construct a rescaled metric

$$\widetilde{g}(t) = \frac{g(t)}{(d_{g(t)}(p_1, p_2))^{1/2}}$$

thus we have that $d_{\widetilde{g}(t)}(p_1, p_2) = 1$. A new theorem shows that

Theorem 3 (Phong-Song-Sturm-Wang) *There exists constant* C *such that* $\frac{1}{C} \le d_{\widetilde{g}(t)}(p_i, p_j) \le C$ *where* $i, j \in \{1, \ldots, k\}$ *with* $i \neq j$. *Additionally we have that* $(\mathbb{S}^2, p, \widetilde{g}(t)) \to (\mathbb{R}^2, \overline{g}(t))$ *where*

$$\overline{g}(t) = \frac{g_{Euclidean}}{\prod_{j=1}^{k-1} |\overrightarrow{x} - p_j|^{2\beta_j}}.$$

Additionally we have some conjectures along with these theorems and results.

Conjecture 1 *1. Potential solution to the Nirenberg problem,*

2. Want to understand Kähler-Ricci flow on singular varieties.

3. Hamilton-Tian conjecture (solved for $\dim \leq 3$ *by Tian-Zhang-Chen-Wang, with supporting result by Bamler).*

If we go up into higher dimensions, we pick a Fano manifold X (where first Chern class $C_1(X) > 0$), and let D be a smooth complex hypersurface such that $[D] = C_1(X)$. Then we have conical singularities along this hypersurface with

$$\frac{\partial g}{\partial t} = -Ric(g) + \lambda g + (1-\lambda)[D]$$

then $R(X)$ is greatest Ricci lower bound.

Conjecture 2 *If* $\lambda \in (0, R(X))$ *then for generic hypersurface* D *and metric* g_0 *this converges to a Kähler-Einstein metric* g_{KE}*. If* $\lambda = R(X)$ *then* $(X, (1-\lambda)D, g(t)) \to (X_\infty, (1-\lambda)D_\infty, g_\infty)$ *where* g_∞ *is Kähler-Einstein. In the case we have* $\lambda \in (R(X), 1)$ *then our result converges to a soliton.*

If we consider complex projective space $\mathbb{CP}^2$ we have a blow-up at one point , where we have $\lambda = R(\mathbb{CP}^2) = \frac{6}{7}$. We have two cases here where a complex hypersurface make develop conical singularities with no multiplicity (only single blow-ups needed), and there are also possible cases where multiple singularities occur where one needs a double blow-up approach.