Convergence of the Kähler-Ricci flow on minimal models

Joint work with P.Eyssidieux and A.Zeriahi

Vincent Guedj

Institut Universitaire de France & Institut de Mathématiques de Toulouse

MSRI, May 5, 2016

Classifying compact Kähler manifolds

Classifying compact Kähler manifolds

Classify algebraic varieties by positivity properties of canonical bundle.

Classifying compact Kähler manifolds

Classify algebraic varieties by positivity properties of canonical bundle.

Definition

Let X be a projective algebraic variety of complex dimension n.

Classifying compact Kähler manifolds

Classify algebraic varieties by positivity properties of canonical bundle.

Definition

Let X be a projective algebraic variety of complex dimension n.
The Kodaira dimension of X is $\kappa := kod(X) \in \{-\infty, 0, 1 \ldots, n\}$, s.t.

Classifying compact Kähler manifolds

Classify algebraic varieties by positivity properties of canonical bundle.

Definition

Let X be a projective algebraic variety of complex dimension n.
The Kodaira dimension of X is $\kappa := kod(X) \in \{-\infty, 0, 1 \dots, n\}$, s.t.

$$dim_{\mathbb{C}} H^0(X, K_X^j) \sim j^{\kappa}.$$

Classifying compact Kähler manifolds

Classify algebraic varieties by positivity properties of canonical bundle.

Definition

Let X be a projective algebraic variety of complex dimension n.
The Kodaira dimension of X is $\kappa := kod(X) \in \{-\infty, 0, 1 \ldots, n\}$, s.t.

$$dim_{\mathbb{C}} H^0(X, K_X^j) \sim j^{\kappa}.$$

- $kod(X)$ is a birational invariant

Classifying compact Kähler manifolds

Classify algebraic varieties by positivity properties of canonical bundle.

Definition

Let X be a projective algebraic variety of complex dimension n.
The Kodaira dimension of X is $\kappa := kod(X) \in \{-\infty, 0, 1 \ldots, n\}$, s.t.

$$dim_{\mathbb{C}} H^0(X, K_X^j) \sim j^{\kappa}.$$

- $kod(X)$ is a birational invariant
- Today always assume $0 \leq kod(X) \leq n$ ("negative Ricci curvature")

Classifying compact Kähler manifolds

Classify algebraic varieties by positivity properties of canonical bundle.

Definition

Let X be a projective algebraic variety of complex dimension n.
The Kodaira dimension of X is $\kappa := kod(X) \in \{-\infty, 0, 1 \ldots, n\}$, s.t.

$$dim_{\mathbb{C}} H^0(X, K_X^j) \sim j^\kappa.$$

- $kod(X)$ is a birational invariant
- Today always assume $0 \leq kod(X) \leq n$ ("negative Ricci curvature")
- Building blocks are Calabi-Yau varieties ($kod(X) = 0$, $mK_X = 0$)

Classifying compact Kähler manifolds

Classify algebraic varieties by positivity properties of canonical bundle.

Definition

Let X be a projective algebraic variety of complex dimension n.
The Kodaira dimension of X is $\kappa := kod(X) \in \{-\infty, 0, 1 \ldots, n\}$, s.t.

$$dim_{\mathbb{C}} H^0(X, K_X^j) \sim j^{\kappa}.$$

- $kod(X)$ is a birational invariant
- Today always assume $0 \leq kod(X) \leq n$ ("negative Ricci curvature")
- Building blocks are Calabi-Yau varieties ($kod(X) = 0$, $mK_X = 0$) and Varieties of general type ($kod(X) = n$, $vol(K_X) > 0$)

Classifying compact Kähler manifolds

Classify algebraic varieties by positivity properties of canonical bundle.

Definition

Let X be a projective algebraic variety of complex dimension n.
The Kodaira dimension of X is $\kappa := kod(X) \in \{-\infty, 0, 1 \ldots, n\}$, s.t.

$$dim_{\mathbb{C}} H^0(X, K_X^j) \sim j^{\kappa}.$$

- $kod(X)$ is a birational invariant
- Today always assume $0 \leq kod(X) \leq n$ ("negative Ricci curvature")
- Building blocks are Calabi-Yau varieties ($kod(X) = 0$, $mK_X = 0$) and Varieties of general type ($kod(X) = n$, $vol(K_X) > 0$)
- Our main focus is on the case $0 < kod(X) < n$ (vol. collapsing)

Studying the Kähler-Ricci flow

Let X be a compact Kähler manifold of complex dimension $n \geq 1$.

Studying the Kähler-Ricci flow

Let X be a compact Kähler manifold of complex dimension $n \geq 1$.
Fix ω_0 a Kähler form and consider the Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) \\ \omega_{|t=0} = \omega_0 \end{cases}$$

Studying the Kähler-Ricci flow

Let X be a compact Kähler manifold of complex dimension $n \geq 1$.
Fix ω_0 a Kähler form and consider the Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) \\ \omega_{|t=0} = \omega_0 \end{cases}$$

This flow admits a unique solution $\omega = \omega(t,x) = \omega_t(x)$ on a maximal domain $[0, T_{max}[\times X$, where

$$T_{max} = \sup\{t > 0\,;\, \{\omega_0\} - tc_1(X) \text{ is Kähler }\}.$$

Studying the Kähler-Ricci flow

Let X be a compact Kähler manifold of complex dimension $n \geq 1$.
Fix ω_0 a Kähler form and consider the Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) \\ \omega_{|t=0} = \omega_0 \end{cases}$$

This flow admits a unique solution $\omega = \omega(t,x) = \omega_t(x)$ on a maximal domain $[0, T_{max}[\times X$, where

$$T_{max} = \sup\{t > 0\,;\, \{\omega_0\} - tc_1(X) \text{ is Kähler }\}.$$

- Thus $T_{max} = +\infty$ iff K_X is nef (smooth minimal model)

Studying the Kähler-Ricci flow

Let X be a compact Kähler manifold of complex dimension $n \geq 1$.
Fix ω_0 a Kähler form and consider the Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) \\ \omega_{|t=0} = \omega_0 \end{cases}$$

This flow admits a unique solution $\omega = \omega(t,x) = \omega_t(x)$ on a maximal domain $[0, T_{max}[\times X$, where

$$T_{max} = \sup\{t > 0\,;\ \{\omega_0\} - tc_1(X) \text{ is Kähler }\}.$$

- Thus $T_{max} = +\infty$ iff K_X is nef (smooth minimal model)
- Volume not collapsing if $kod(X) = n$ (and $kod(X) = 0$).

An ambitious program [Song-Tian]

A natural and difficult problem is to understand the asymptotic behavior of ω_t as $t \to T_{max}$. Ideally one would like to

An ambitious program [Song-Tian]

A natural and difficult problem is to understand the asymptotic behavior of ω_t as $t \to T_{max}$. Ideally one would like to

- show that (X, ω_t) converges to a midly singular Kähler variety (X_1, S_1) equipped with a singular *Kähler current* S_1;

An ambitious program [Song-Tian]

A natural and difficult problem is to understand the asymptotic behavior of ω_t as $t \to T_{max}$. Ideally one would like to

- show that (X, ω_t) converges to a midly singular Kähler variety (X_1, S_1) equipped with a singular *Kähler current* S_1;
- try and restart the KRF on X_1 with initial data S_1;

An ambitious program [Song-Tian]

A natural and difficult problem is to understand the asymptotic behavior of ω_t as $t \to T_{max}$. Ideally one would like to

- show that (X, ω_t) converges to a midly singular Kähler variety (X_1, S_1) equipped with a singular *Kähler current* S_1;
- try and restart the KRF on X_1 with initial data S_1;
- repeat finitely many times to reach a minimal model X_r;

An ambitious program [Song-Tian]

A natural and difficult problem is to understand the asymptotic behavior of ω_t as $t \to T_{max}$. Ideally one would like to

- show that (X, ω_t) converges to a midly singular Kähler variety (X_1, S_1) equipped with a singular *Kähler current* S_1;
- try and restart the KRF on X_1 with initial data S_1;
- repeat finitely many times to reach a minimal model X_r;
- study the long term behavior of the NKRF (K_{X_r} is *nef*),

$$\begin{cases} \frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) - \omega_\mathrm{t} \\ \omega_{|t=0} = S_r \end{cases}$$

and show that (X_r, ω_t) converges to a canonical model (X_{can}, ω_{can}).

Known results

- Program achieved in dimension one, Hamilton [1986] & Chow [1991].

Known results

- Program achieved in dimension one, Hamilton [1986] & Chow [1991].
- More or less complete in dimension two (...Song-Weinkove [2013]).

Known results

- Program achieved in dimension one, Hamilton [1986] & Chow [1991].
- More or less complete in dimension two (...Song-Weinkove [2013]).
- Program however largely open in dimension ≥ 3.

Known results

- Program achieved in dimension one, Hamilton [1986] & Chow [1991].
- More or less complete in dimension two (...Song-Weinkove [2013]).
- Program however largely open in dimension ≥ 3.
- Many difficulties to overcome, among them

Known results

- Program achieved in dimension one, Hamilton [1986] & Chow [1991].
- More or less complete in dimension two (...Song-Weinkove [2013]).
- Program however largely open in dimension ≥ 3.
- Many difficulties to overcome, among them
  - Degenerate initial data (Kähler current rather than a Kähler form).

Known results

- Program achieved in dimension one, Hamilton [1986] & Chow [1991].
- More or less complete in dimension two (...Song-Weinkove [2013]).
- Program however largely open in dimension ≥ 3.
- Many difficulties to overcome, among them
  - Degenerate initial data (Kähler current rather than a Kähler form).
    OK by works of ...[Song-Tian09], [GZ13], [DiNezza-Lu14].

Known results

- Program achieved in dimension one, Hamilton [1986] & Chow [1991].
- More or less complete in dimension two (...Song-Weinkove [2013]).
- Program however largely open in dimension ≥ 3.
- Many difficulties to overcome, among them
  - Degenerate initial data (Kähler current rather than a Kähler form).
    OK by works of ...[Song-Tian09], [GZ13], [DiNezza-Lu14].
  - Define the KRF on mildly singular varieties.

Known results

- Program achieved in dimension one, Hamilton [1986] & Chow [1991].
- More or less complete in dimension two (...Song-Weinkove [2013]).
- Program however largely open in dimension ≥ 3.
- Many difficulties to overcome, among them
  - Degenerate initial data (Kähler current rather than a Kähler form).
    OK by works of ...[Song-Tian09], [GZ13], [DiNezza-Lu14].
  - Define the KRF on mildly singular varieties.
    OK by works of ...[Song-Tian09], [EGZ14].

Known results

- Program achieved in dimension one, Hamilton [1986] & Chow [1991].
- More or less complete in dimension two (...Song-Weinkove [2013]).
- Program however largely open in dimension ≥ 3.
- Many difficulties to overcome, among them
  - Degenerate initial data (Kähler current rather than a Kähler form).
    OK by works of ...[Song-Tian09], [GZ13], [DiNezza-Lu14].
  - Define the KRF on mildly singular varieties.
    OK by works of ...[Song-Tian09], [EGZ14].
  - Construct canonical limits and prove convergence.

Known results

- Program achieved in dimension one, Hamilton [1986] & Chow [1991].
- More or less complete in dimension two (...Song-Weinkove [2013]).
- Program however largely open in dimension ≥ 3.
- Many difficulties to overcome, among them
  - Degenerate initial data (Kähler current rather than a Kähler form).
    OK by works of ...[Song-Tian09], [GZ13], [DiNezza-Lu14].
  - Define the KRF on mildly singular varieties.
    OK by works of ...[Song-Tian09], [EGZ14].
  - Construct canonical limits and prove convergence.
    $\longrightarrow$ *today's lecture.*

Known results

- Cv on $\mathbb{Q}$-Calabi-Yau varieties [Cao85, Song-Tian09, EGZ14]

Known results

- Cv on $\mathbb{Q}$-Calabi-Yau varieties [Cao85, Song-Tian09, EGZ14]
- Cv on general type minimal models [Tsuji88, Tian-Zhang06, EGZ14]

Known results

- Cv on $\mathbb{Q}$-Calabi-Yau varieties [Cao85, Song-Tian09, EGZ14]
- Cv on general type minimal models [Tsuji88, Tian-Zhang06, EGZ14]
- Cv on smooth minimal models of intermediate Kodaira dimension [Song-Tian07, Song-Tian12, Tosatti-Weinkove-Yang14, etc]

Known results

- Cv on $\mathbb{Q}$-Calabi-Yau varieties [Cao85, Song-Tian09, EGZ14]
- Cv on general type minimal models [Tsuji88, Tian-Zhang06, EGZ14]
- Cv on smooth minimal models of intermediate Kodaira dimension [Song-Tian07, Song-Tian12, Tosatti-Weinkove-Yang14, etc]
- Cv on arbitrary minimal models of intermediate Kodaira dimension :

Known results

- Cv on $\mathbb{Q}$-Calabi-Yau varieties [Cao85, Song-Tian09, EGZ14]
- Cv on general type minimal models [Tsuji88, Tian-Zhang06, EGZ14]
- Cv on smooth minimal models of intermediate Kodaira dimension [Song-Tian07, Song-Tian12, Tosatti-Weinkove-Yang14, etc]
- Cv on arbitrary minimal models of intermediate Kodaira dimension :

Theorem (EGZ16)

Let X be a 3-dim. minimal model with canonical singularities. The NKRF continuously deforms any Kähler form ω_0 to a canonical current T_{can}.

Known results

- Cv on $\mathbb{Q}$-Calabi-Yau varieties [Cao85, Song-Tian09, EGZ14]
- Cv on general type minimal models [Tsuji88, Tian-Zhang06, EGZ14]
- Cv on smooth minimal models of intermediate Kodaira dimension [Song-Tian07, Song-Tian12, Tosatti-Weinkove-Yang14, etc]
- Cv on arbitrary minimal models of intermediate Kodaira dimension :

Theorem (EGZ16)

Let X be a 3-dim. minimal model with canonical singularities. The NKRF continuously deforms any Kähler form ω_0 to a canonical current T_{can}.

Main ingredient=viscosity methods [EGZ16]

Mild singularities

- Singularities showing up in the Minimal Model Program

Mild singularities

- Singularities showing up in the Minimal Model Program
- Sufficient for our purpose to deal with *canonical singularities*,

Mild singularities

- Singularities showing up in the Minimal Model Program
- Sufficient for our purpose to deal with *canonical singularities*, i.e. X is $\mathbb{Q}$-Gorenstein of finite index and for any resolution,

Mild singularities

- Singularities showing up in the Minimal Model Program
- Sufficient for our purpose to deal with *canonical singularities*, i.e. X is $\mathbb{Q}$-Gorenstein of finite index and for any resolution,

$$K_{\tilde{X}} = \pi^* K_X + \sum_i a_i E_i \text{ with } a_i \geq 0$$

Mild singularities

- Singularities showing up in the Minimal Model Program
- Sufficient for our purpose to deal with *canonical singularities*, i.e. X is $\mathbb{Q}$-Gorenstein of finite index and for any resolution,

$$K_{\tilde{X}} = \pi^* K_X + \sum_i a_i E_i \text{ with } a_i \geq 0$$

- Example : $\sum_{j=0}^{n} z_j^2 = 0 \longleftrightarrow$ the ordinary double point.

Mild singularities

- Singularities showing up in the Minimal Model Program
- Sufficient for our purpose to deal with *canonical singularities*, i.e. X is $\mathbb{Q}$-Gorenstein of finite index and for any resolution,

$$K_{\tilde{X}} = \pi^* K_X + \sum_i a_i E_i \text{ with } a_i \geq 0$$

- Example : $\sum_{j=0}^{n} z_j^2 = 0 \longleftrightarrow$ the ordinary double point.
- This is not a quotient singularity if $n \geq 3$.

Complex Monge-Ampère flows

Solving the normalized Kähler-Ricci flow is equivalent to solving

$$(CMAF) \qquad (\omega_t + dd^c \varphi_t)^n = e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x),$$

Complex Monge-Ampère flows

Solving the normalized Kähler-Ricci flow is equivalent to solving

$$(CMAF) \qquad (\omega_t + dd^c \varphi_t)^n = e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x),$$

- $t \mapsto \omega_t(x)$ continuous family of semi-positive closed $(1,1)$-forms;

Complex Monge-Ampère flows

Solving the normalized Kähler-Ricci flow is equivalent to solving

$$(CMAF) \qquad (\omega_t + dd^c \varphi_t)^n = e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x),$$

- $t \mapsto \omega_t(x)$ continuous family of semi-positive closed $(1,1)$-forms;
- $(t,x) \mapsto h(t,x)$ is continuous

Complex Monge-Ampère flows

Solving the normalized Kähler-Ricci flow is equivalent to solving

$$(CMAF) \qquad (\omega_t + dd^c \varphi_t)^n = e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x),$$

- $t \mapsto \omega_t(x)$ continuous family of semi-positive closed $(1,1)$-forms;
- $(t,x) \mapsto h(t,x)$ is continuous
- ψ is quasi-psh and continuous (i.e. e^{ψ} is continuous),

Complex Monge-Ampère flows

Solving the normalized Kähler-Ricci flow is equivalent to solving

$$(CMAF) \qquad (\omega_t + dd^c \varphi_t)^n = e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x),$$

- $t \mapsto \omega_t(x)$ continuous family of semi-positive closed $(1,1)$-forms;
- $(t,x) \mapsto h(t,x)$ is continuous
- ψ is quasi-psh and continuous (i.e. e^{ψ} is continuous),

and $(t,x) \mapsto \varphi(t,x) = \varphi_t(x)$ is the unknown function.

Degeneracies

We work on a smooth manifold $\tilde{X}$, obtained by desingularizing a singular model: if $\pi : \tilde{X} \to X$ denotes a resolution of singularities, then

Degeneracies

We work on a smooth manifold $\tilde{X}$, obtained by desingularizing a singular model: if $\pi : \tilde{X} \to X$ denotes a resolution of singularities, then

- $\omega_t = \pi^* \theta_t$ are pull-backs of Kähler forms hence no longer Kähler;

Degeneracies

We work on a smooth manifold $\tilde{X}$, obtained by desingularizing a singular model: if $\pi : \tilde{X} \to X$ denotes a resolution of singularities, then

- $\omega_t = \pi^* \theta_t$ are pull-backs of Kähler forms hence no longer Kähler;
- the RHS vanishes along the exceptional divisors, e.g.

Degeneracies

We work on a smooth manifold $\tilde{X}$, obtained by desingularizing a singular model: if $\pi : \tilde{X} \to X$ denotes a resolution of singularities, then

- $\omega_t = \pi^* \theta_t$ are pull-backs of Kähler forms hence no longer Kähler;
- the RHS vanishes along the exceptional divisors, e.g.

$$e^{\psi} = \prod_{j=1}^{N} |s_j|_h^2 \longleftrightarrow \text{canonical singularities.}$$

Existence result

We always assume that ω_t is "regular", i.e.

Existence result

We always assume that ω_t is "regular", i.e.

- $\exists \theta$ semi-positive and big s.t. $\theta \leq \omega_t$ for all t;

Existence result

We always assume that ω_t is "regular", i.e.

- $\exists \theta$ semi-positive and big s.t. $\theta \leq \omega_t$ for all t;
- $\exists \varepsilon \in \mathcal{C}^1$ with $\varepsilon(0) = 0$ s.t. $\omega_s(1 - \varepsilon(t-s)) \leq \omega_t$.

Existence result

We always assume that ω_t is "regular", i.e.

- $\exists \theta$ semi-positive and big s.t. $\theta \leq \omega_t$ for all t;
- $\exists \varepsilon \in \mathcal{C}^1$ with $\varepsilon(0) = 0$ s.t. $\omega_s(1 - \varepsilon(t-s)) \leq \omega_t$.

Theorem (EGZ14)

If φ_0 is an arbitrary continuous ω_0-psh function, there exists a unique viscosity solution $(t,x) \mapsto \varphi_t(x)$ of (CMAF) with initial value φ_0.

Existence result

We always assume that ω_t is "regular", i.e.

- $\exists\theta$ semi-positive and big s.t. $\theta \leq \omega_t$ for all t;
- $\exists\varepsilon \in \mathcal{C}^1$ with $\varepsilon(0) = 0$ s.t. $\omega_s(1 - \varepsilon(t - s)) \leq \omega_t$.

Theorem (EGZ14)

If φ_0 is an arbitrary continuous ω_0-psh function, there exists a unique viscosity solution $(t, x) \mapsto \varphi_t(x)$ of (CMAF) with initial value φ_0. The function φ_t is the upper envelope of viscosity subsolutions. In particular $x \mapsto \varphi_t(x)$ is ω_t-plurisubharmonic for all $t \geq 0$.

Existence result

We always assume that ω_t is "regular", i.e.

- $\exists \theta$ semi-positive and big s.t. $\theta \leq \omega_t$ for all t;
- $\exists \varepsilon \in \mathcal{C}^1$ with $\varepsilon(0) = 0$ s.t. $\omega_s(1 - \varepsilon(t-s)) \leq \omega_t$.

Theorem (EGZ14)

If φ_0 is an arbitrary continuous ω_0-psh function, there exists a unique viscosity solution $(t, x) \mapsto \varphi_t(x)$ of (CMAF) with initial value φ_0. The function φ_t is the upper envelope of viscosity subsolutions. In particular $x \mapsto \varphi_t(x)$ is ω_t-plurisubharmonic for all $t \geq 0$.

NB: need extension of this result to manifolds with boundary [EGZ16]

Classical sub/super/solutions

Definition

A function $\varphi \in \mathcal{C}^{1,2}$ *is a* **classical subsolution** *of (CMAF) if for all* $t \geq 0$ $x \mapsto \varphi_t(x)$ *is* ω_t*-psh and*

$$(\omega_t + dd^c \varphi_t)^n \geq e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x)$$

Classical sub/super/solutions

Definition

A function $\varphi \in \mathcal{C}^{1,2}$ *is a* **classical subsolution** *of (CMAF) if for all* $t \geq 0$ $x \mapsto \varphi_t(x)$ *is* ω_t*-psh and*

$$(\omega_t + dd^c \varphi_t)^n \geq e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x)$$

A function $\varphi \in \mathcal{C}^{1,2}$ *is a* **classical supersolution** *of (CMAF) if*

$$(\omega_t + dd^c \varphi_t)_+^n \leq e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x)$$

Classical sub/super/solutions

Definition

A function $\varphi \in \mathcal{C}^{1,2}$ is a **classical subsolution** *of (CMAF) if for all $t \geq 0$ $x \mapsto \varphi_t(x)$ is ω_t-psh and*

$$(\omega_t + dd^c \varphi_t)^n \geq e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x)$$

A function $\varphi \in \mathcal{C}^{1,2}$ is a **classical supersolution** *of (CMAF) if*

$$(\omega_t + dd^c \varphi_t)_+^n \leq e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x)$$

Here $\theta_+(x) = \theta(x)$ if $\theta(x) \geq 0$ and $\theta_+(x) = 0$ otherwise.

Classical sub/super/solutions

Definition

A function $\varphi \in \mathcal{C}^{1,2}$ *is a* **classical subsolution** *of (CMAF) if for all* $t \geq 0$ $x \mapsto \varphi_t(x)$ *is* ω_t*-psh and*

$$(\omega_t + dd^c \varphi_t)^n \geq e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x)$$

A function $\varphi \in \mathcal{C}^{1,2}$ *is a* **classical supersolution** *of (CMAF) if*

$$(\omega_t + dd^c \varphi_t)_+^n \leq e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x)$$

A **classical solution** *is both a subsolution and a supersolution.*

Here $\theta_+(x) = \theta(x)$ if $\theta(x) \geq 0$ and $\theta_+(x) = 0$ otherwise.

Classical sub/super/solutions

Definition

A function $\varphi \in \mathcal{C}^{1,2}$ *is a* **classical subsolution** *of (CMAF) if for all* $t \geq 0$ $x \mapsto \varphi_t(x)$ *is* ω_t*-psh and*

$$(\omega_t + dd^c \varphi_t)^n \geq e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x)$$

A function $\varphi \in \mathcal{C}^{1,2}$ *is a* **classical supersolution** *of (CMAF) if*

$$(\omega_t + dd^c \varphi_t)_+^n \leq e^{\dot{\varphi}_t + \varphi_t + h(t,x)} e^{\psi(x)} dV(x)$$

A **classical solution** *is both a subsolution and a supersolution.*

Here $\theta_+(x) = \theta(x)$ if $\theta(x) \geq 0$ and $\theta_+(x) = 0$ otherwise.

PROBLEM: classical solutions usually do not exist !

Viscosity subsolutions

Definition

Given $u : X_T := (0, T) \times \tilde{X} \to \mathbb{R}$ *an u.s.c. bounded function and* $(t_0, x_0) \in X_T$, q *is a differential test from above for* u *at* (t_0, x_0) *if*

Viscosity subsolutions

Definition

Given $u : X_T := (0, T) \times \tilde{X} \to \mathbb{R}$ an u.s.c. bounded function and $(t_0, x_0) \in X_T$, q is a differential test from above for u at (t_0, x_0) if

- *$q \in \mathcal{C}^{1,2}$ in a small neighborhood V_0 of (t_0, x_0);*

Viscosity subsolutions

Definition

Given $u : X_T := (0, T) \times \tilde{X} \to \mathbb{R}$ *an u.s.c. bounded function and* $(t_0, x_0) \in X_T$, q *is a differential test from above for* u *at* (t_0, x_0) *if*

- $q \in \mathcal{C}^{1,2}$ *in a small neighborhood* V_0 *of* (t_0, x_0);
- $u \leq q$ *in* V_0 *and* $u(t_0, x_0) = q(t_0, x_0)$.

Viscosity subsolutions

Definition

Given $u : X_T := (0, T) \times \tilde{X} \to \mathbb{R}$ *an u.s.c. bounded function and* $(t_0, x_0) \in X_T$*,* q *is a differential test from above for* u *at* (t_0, x_0) *if*

- $q \in \mathcal{C}^{1,2}$ *in a small neighborhood* V_0 *of* (t_0, x_0)*;*
- $u \leq q$ *in* V_0 *and* $u(t_0, x_0) = q(t_0, x_0)$*.*

Definition

An u.s.c. bounded function $u : X_T \to \mathbb{R}$ *is a* **viscosity subsolution** *of (CMAF) if for all* $(t_0, x_0) \in X_T$ *and all differential tests* q *from above,*

$$(\omega_{t_0}(x_0) + dd^c q_{t_0}(x_0))^n \geq e^{\dot{q}_{t_0}(x_0) + q_{t_0}(x_0) + h(t_0, x_0)} e^{\psi(x_0)} dV(x_0).$$

Viscosity super/solutions

Definition

A l.s.c. bounded function $v : X_T \to \mathbb{R}$ *is a* **viscosity supersolution** *of (CMAF) if for all* $(t_0, x_0) \in X_T$ *and all differential tests* q *from below,*

$$(\omega_{t_0}(x_0) + dd^c q_{t_0}(x_0))_+^n \leq e^{\dot{q}_{t_0}(x_0) + q_{t_0}(x_0) + h(t_0, x_0)} e^{\psi(x_0)} dV(x_0).$$

Viscosity super/solutions

Definition

A l.s.c. bounded function $v : X_T \to \mathbb{R}$ is a **viscosity supersolution** *of (CMAF) if for all $(t_0, x_0) \in X_T$ and all differential tests q from below,*

$$(\omega_{t_0}(x_0) + dd^c q_{t_0}(x_0))_+^n \leq e^{\dot{q}_{t_0}(x_0) + q_{t_0}(x_0) + h(t_0, x_0)} e^{\psi(x_0)} dV(x_0).$$

Definition

A **viscosity solution** *of (CMAF) is a continuous function which is both a viscosity subsolution and a viscosity supersolution.*

Basic facts

- Assume u is $\mathcal{C}^{1,2}$-smooth. It is a viscosity subsolution iff it is ω_t-psh and a classical subsolution (similar result for supersolution).

Basic facts

- Assume u is $\mathcal{C}^{1,2}$-smooth. It is a viscosity subsolution iff it is ω_t-psh and a classical subsolution (similar result for supersolution).
- If u_1, u_2 are viscosity subsolutions, then so is $\max(u_1, u_2)$.

Basic facts

- Assume u is $\mathcal{C}^{1,2}$-smooth. It is a viscosity subsolution iff it is ω_t-psh and a classical subsolution (similar result for supersolution).
- If u_1, u_2 are viscosity subsolutions, then so is $\max(u_1, u_2)$.
- If $(u_\alpha)_{\alpha \in A}$ is a loc. unif. bdd above family of subsolutions, then

$$\varphi := (\sup\{u_\alpha, \ \alpha \in A\})^* \text{ is a subsolution.}$$

Basic facts

- Assume u is $\mathcal{C}^{1,2}$-smooth. It is a viscosity subsolution iff it is ω_t-psh and a classical subsolution (similar result for supersolution).
- If u_1, u_2 are viscosity subsolutions, then so is $\max(u_1, u_2)$.
- If $(u_\alpha)_{\alpha \in A}$ is a loc. unif. bdd above family of subsolutions, then

$$\varphi := (\sup\{u_\alpha,\ \alpha \in A\})^* \text{ is a subsolution.}$$

- If u is a subsolution of $(CMAF)_\mu$, where $\mu := e^{h+\psi} dV$, then it is also a subsolution of $(CMAF)_\nu$ for all $0 \leq \nu \leq \mu$.

Basic facts

- Assume u is $\mathcal{C}^{1,2}$-smooth. It is a viscosity subsolution iff it is ω_t-psh and a classical subsolution (similar result for supersolution).
- If u_1, u_2 are viscosity subsolutions, then so is $\max(u_1, u_2)$.
- If $(u_\alpha)_{\alpha \in A}$ is a loc. unif. bdd above family of subsolutions, then

$$\varphi := (\sup\{u_\alpha,\ \alpha \in A\})^* \text{ is a subsolution.}$$

- If u is a subsolution of $(CMAF)_\mu$, where $\mu := e^{h+\psi} dV$, then it is also a subsolution of $(CMAF)_\nu$ for all $0 \leq \nu \leq \mu$.
- u is a subsolution of $(CMAF)_0$ iff $x \mapsto u_t(x)$ is ω_t-psh $\forall t \geq 0$.

The comparison principle

The key result here is the following maximum principle:

The comparison principle

The key result here is the following maximum principle:

Theorem (EGZ14+16)

Assume u is a subsolution to (CMAF) and

The comparison principle

The key result here is the following maximum principle:

Theorem (EGZ14+16)

Assume u is a subsolution to (CMAF) and
v is a supersolution to (CMAF).

The comparison principle

The key result here is the following maximum principle:

Theorem (EGZ14+16)

Assume u is a subsolution to (CMAF) and
v is a supersolution to (CMAF).
Then $u_0 \leq v_0 \Longrightarrow u_t \leq v_t$ for all $t > 0$.

The comparison principle

The key result here is the following maximum principle:

Theorem (EGZ14+16)

Assume u is a subsolution to (CMAF) and
v is a supersolution to (CMAF).
Then $u_0 \leq v_0 \Longrightarrow u_t \leq v_t$ for all $t > 0$.

$\rightarrow$ implies uniqueness of solutions.

The comparison principle

The key result here is the following maximum principle:

Theorem (EGZ14+16)

Assume u is a subsolution to (CMAF) and
v is a supersolution to (CMAF).
Then $u_0 \leq v_0 \Longrightarrow u_t \leq v_t$ for all $t > 0$.

$\rightarrow$ implies uniqueness of solutions.
$\rightarrow$ the key to the existence of solutions.

The canonical twisted Kähler-Einstein current

Let X be an abundant minimal model with canonical singularities:

The canonical twisted Kähler-Einstein current

Let X be an abundant minimal model with canonical singularities:
K_X is a semi-ample $\mathbb{Q}$-line bundle, $\kappa =$Kodaira dimension of X, and

The canonical twisted Kähler-Einstein current

Let X be an abundant minimal model with canonical singularities:
K_X is a semi-ample $\mathbb{Q}$-line bundle, κ =Kodaira dimension of X, and

- $f : X \to X_{can}$ = Iitaka fibration, A ample $\mathbb{Q}$-line bdle s.t. $K_X = f^*A$.

The canonical twisted Kähler-Einstein current

Let X be an abundant minimal model with canonical singularities:
K_X is a semi-ample $\mathbb{Q}$-line bundle, κ =Kodaira dimension of X, and

- $f : X \to X_{can}$ = Iitaka fibration, A ample $\mathbb{Q}$-line bdle s.t. $K_X = f^*A$.
- Generic fiber $X_y = f^{-1}(y)$ is a $\mathbb{Q}$-Calabi-Yau variety.

The canonical twisted Kähler-Einstein current

Let X be an abundant minimal model with canonical singularities: K_X is a semi-ample $\mathbb{Q}$-line bundle, $\kappa =$Kodaira dimension of X, and

- $f : X \to X_{can} =$ Iitaka fibration, A ample $\mathbb{Q}$-line bdle s.t. $K_X = f^*A$.
- Generic fiber $X_y = f^{-1}(y)$ is a $\mathbb{Q}$-Calabi-Yau variety.
- Fix h_A a positive hermitian metric of A with curvature form ω_A.

The canonical twisted Kähler-Einstein current

Let X be an abundant minimal model with canonical singularities:
K_X is a semi-ample $\mathbb{Q}$-line bundle, κ =Kodaira dimension of X, and

- $f : X \to X_{can}$ = Iitaka fibration, A ample $\mathbb{Q}$-line bdle s.t. $K_X = f^*A$.
- Generic fiber $X_y = f^{-1}(y)$ is a $\mathbb{Q}$-Calabi-Yau variety.
- Fix h_A a positive hermitian metric of A with curvature form ω_A.
- Fix η local (multivalued) non-vanishing hol. section of K_X, $\tilde{h}_A = f^*h_A$

The canonical twisted Kähler-Einstein current

Let X be an abundant minimal model with canonical singularities: K_X is a semi-ample $\mathbb{Q}$-line bundle, $\kappa =$Kodaira dimension of X, and

- $f : X \to X_{can} =$ Iitaka fibration, A ample $\mathbb{Q}$-line bdle s.t. $K_X = f^*A$.
- Generic fiber $X_y = f^{-1}(y)$ is a $\mathbb{Q}$-Calabi-Yau variety.
- Fix h_A a positive hermitian metric of A with curvature form ω_A.
- Fix η local (multivalued) non-vanishing hol. section of K_X, $\tilde{h}_A = f^*h_A$
- and $v(h_A) = c_n \frac{\eta \wedge \overline{\eta}}{||\eta||^2_{\tilde{h}_A}} =$ globally well defined volume form on X.

The canonical twisted Kähler-Einstein current

Let X be an abundant minimal model with canonical singularities: K_X is a semi-ample $\mathbb{Q}$-line bundle, κ =Kodaira dimension of X, and

- $f : X \to X_{can}$ = Iitaka fibration, A ample $\mathbb{Q}$-line bdle s.t. $K_X = f^*A$.
- Generic fiber $X_y = f^{-1}(y)$ is a $\mathbb{Q}$-Calabi-Yau variety.
- Fix h_A a positive hermitian metric of A with curvature form ω_A.
- Fix η local (multivalued) non-vanishing hol. section of K_X, $\tilde{h}_A = f^*h_A$
- and $v(h_A) = c_n \frac{\eta \wedge \overline{\eta}}{||\eta||^2_{\tilde{h}_A}}$ = globally well defined volume form on X.

Lemma

The measure $f_* v(h_A)$ has density in $L^{1+\varepsilon}$ w.r.t to ω_A^κ.

The canonical twisted Kähler-Einstein current

Theorem (EGZ11, EGZ16)

There exists a unique continuous ω_A-psh function φ_{can} on X_{can} s.t.

$$(\omega_A + dd^c \varphi_{can})^\kappa = e^{\varphi_{can}} f_*(v(h_A)).$$

The canonical twisted Kähler-Einstein current

Theorem (EGZ11, EGZ16)

There exists a unique continuous ω_A-psh function φ_{can} on X_{can} s.t.

$$(\omega_A + dd^c \varphi_{can})^{\kappa} = e^{\varphi_{can}} f_*(v(h_A)).$$

- *The current $\omega_{can} = \omega_A + dd^c \varphi_{can}$ is independent of h_A.*

The canonical twisted Kähler-Einstein current

Theorem (EGZ11, EGZ16)

There exists a unique continuous ω_A-psh function φ_{can} on X_{can} s.t.

$$(\omega_A + dd^c \varphi_{can})^{\kappa} = e^{\varphi_{can}} f_*(v(h_A)).$$

- *The current $\omega_{can} = \omega_A + dd^c \varphi_{can}$ is independent of h_A.*
- *It is smooth in $X_{can}^{reg} \setminus \text{critical values of f}$.*

The canonical twisted Kähler-Einstein current

Theorem (EGZ11, EGZ16)

There exists a unique continuous ω_A-psh function φ_{can} on X_{can} s.t.

$$(\omega_A + dd^c \varphi_{can})^\kappa = e^{\varphi_{can}} f_*(v(h_A)).$$

- *The current $\omega_{can} = \omega_A + dd^c \varphi_{can}$ is independent of h_A.*
- *It is smooth in $X_{can}^{reg} \setminus \text{critical values of f}$.*
- *It satisfies $\mathrm{Ric}(\omega_{\mathrm{can}}) = -\omega_{\mathrm{can}} + \omega_{\mathrm{WP}}$ in $X_{can}^{reg} \setminus \text{critical values}$.*

The canonical twisted Kähler-Einstein current

Theorem (EGZ11, EGZ16)

There exists a unique continuous ω_A-psh function φ_{can} on X_{can} s.t.

$$(\omega_A + dd^c \varphi_{can})^\kappa = e^{\varphi_{can}} f_*(v(h_A)).$$

- *The current $\omega_{can} = \omega_A + dd^c \varphi_{can}$ is independent of h_A.*
- *It is smooth in $X_{can}^{reg} \setminus \text{critical values of f}$.*
- *It satisfies $\mathrm{Ric}(\omega_{\mathrm{can}}) = -\omega_{\mathrm{can}} + \omega_{\mathrm{WP}}$ in $X_{can}^{reg} \setminus \text{critical values}$.*

- Result due to Song-Tian [ST07,ST12] when X is smooth.

The canonical twisted Kähler-Einstein current

Theorem (EGZ11, EGZ16)

There exists a unique continuous ω_A-psh function φ_{can} on X_{can} s.t.

$$(\omega_A + dd^c \varphi_{can})^{\kappa} = e^{\varphi_{can}} f_*(v(h_A)).$$

- *The current $\omega_{can} = \omega_A + dd^c \varphi_{can}$ is independent of h_A.*
- *It is smooth in $X_{can}^{reg} \setminus \text{critical values of f}$.*
- *It satisfies $\mathrm{Ric}(\omega_{\mathrm{can}}) = -\omega_{\mathrm{can}} + \omega_{\mathrm{WP}}$ in $X_{can}^{reg} \setminus \text{critical values}$.*

- Result due to Song-Tian [ST07,ST12] when X is smooth.
- Weil-Petersson metric $\leftrightarrow$ variation of cplx structure of CY fibers.

The canonical twisted Kähler-Einstein current

The current $T_{can} = f^* \omega_{can}$ is an important birational invariant s.t.

The canonical twisted Kähler-Einstein current

The current $T_{can} = f^*\omega_{can}$ is an important birational invariant s.t.

Lemma (ST12)

$$T_{can}^{\kappa} \wedge \omega_{SF}^{n-\kappa} = e^{\varphi_{can} \circ f} v(h_A).$$

The canonical twisted Kähler-Einstein current

The current $T_{can} = f^*\omega_{can}$ is an important birational invariant s.t.

Lemma (ST12)

$$T_{can}^{\kappa} \wedge \omega_{SF}^{n-\kappa} = e^{\varphi_{can} \circ f} v(h_A).$$

Here $\omega_{SF} = \omega_0 + dd^c \rho$=fiberwise family of Ricci flat KE metrics

The canonical twisted Kähler-Einstein current

The current $T_{can} = f^* \omega_{can}$ is an important birational invariant s.t.

Lemma (ST12)

$$T_{can}^{\kappa} \wedge \omega_{SF}^{n-\kappa} = e^{\varphi_{can} \circ f} v(h_A).$$

Here $\omega_{SF} = \omega_0 + dd^c \rho$=fiberwise family of Ricci flat KE metrics

$$\omega_{SF|X_y} = \text{unique Ricci flat metric in } \{\omega_0\}_{|X_y} \quad [EGZ09]$$

The canonical twisted Kähler-Einstein current

The current $T_{can} = f^*\omega_{can}$ is an important birational invariant s.t.

Lemma (ST12)

$$T_{can}^{\kappa} \wedge \omega_{SF}^{n-\kappa} = e^{\varphi_{can} \circ f} v(h_A).$$

Here $\omega_{SF} = \omega_0 + dd^c \rho$=fiberwise family of Ricci flat KE metrics

$$\omega_{SF|X_y} = \text{unique Ricci flat metric in } \{\omega_0\}_{|X_y} \quad [EGZ09]$$

Conjecture (EGZ16)

The function ρ is smooth in a Zariski open set.

The canonical twisted Kähler-Einstein current

The current $T_{can} = f^*\omega_{can}$ is an important birational invariant s.t.

Lemma (ST12)

$$T_{can}^{\kappa} \wedge \omega_{SF}^{n-\kappa} = e^{\varphi_{can} \circ f} v(h_A).$$

Here $\omega_{SF} = \omega_0 + dd^c \rho$=fiberwise family of Ricci flat KE metrics

$$\omega_{SF|X_y} = \text{unique Ricci flat metric in } \{\omega_0\}_{|X_y} \quad [EGZ09]$$

Conjecture (EGZ16)

The function ρ is smooth in a Zariski open set.

- OK when X is smooth [Yau78] or when $\dim_{\mathbb{C}} X = 3$.

The canonical twisted Kähler-Einstein current

The current $T_{can} = f^*\omega_{can}$ is an important birational invariant s.t.

Lemma (ST12)

$$T_{can}^{\kappa} \wedge \omega_{SF}^{n-\kappa} = e^{\varphi_{can} \circ f} v(h_A).$$

Here $\omega_{SF} = \omega_0 + dd^c \rho$=fiberwise family of Ricci flat KE metrics

$$\omega_{SF|X_y} = \text{unique Ricci flat metric in } \{\omega_0\}_{|X_y} \quad [EGZ09]$$

Conjecture (EGZ16)

The function ρ is smooth in a Zariski open set.

- OK when X is smooth [Yau78] or when $\dim_{\mathbb{C}} X = 3$.
- [Choi15 ?] : the function ρ is ω_0-psh in all variables (i.e. $\omega_{SF} \geq 0$)

The normalized Kähler-Ricci flow

The normalized Kähler-Ricci flow on X can be written as

$$\frac{(\omega_t + dd^c\varphi_t)^n}{C_n^\kappa e^{-(n-\kappa)t}} = e^{\dot{\varphi}_t+\varphi_t} v(h_A).$$

The normalized Kähler-Ricci flow

The normalized Kähler-Ricci flow on X can be written as

$$\frac{(\omega_t + dd^c \varphi_t)^n}{C_n^\kappa e^{-(n-\kappa)t}} = e^{\dot{\varphi}_t + \varphi_t} v(h_A).$$

Particular case of (CMAF) considered above with $T = +\infty$, $h \equiv 0$ and

The normalized Kähler-Ricci flow

The normalized Kähler-Ricci flow on X can be written as

$$\frac{(\omega_t + dd^c\varphi_t)^n}{C_n^\kappa e^{-(n-\kappa)t}} = e^{\dot{\varphi}_t+\varphi_t} v(h_A).$$

Particular case of (CMAF) considered above with $T = +\infty$, $h \equiv 0$ and

- $e^\psi dV = \pi^* v(h_A)$ if $\pi : \tilde{X} \to X$ =desingularization of X;

The normalized Kähler-Ricci flow

The normalized Kähler-Ricci flow on X can be written as

$$\frac{(\omega_t + dd^c \varphi_t)^n}{C_n^\kappa e^{-(n-\kappa)t}} = e^{\dot{\varphi}_t + \varphi_t} v(h_A).$$

Particular case of (CMAF) considered above with $T = +\infty$, $h \equiv 0$ and

- $e^\psi dV = \pi^* v(h_A)$ if $\pi : \tilde{X} \to X$ =desingularization of X;
- $\omega_t = e^{-t}\omega_0 + (1 - e^{-t}) f^* \omega_A$ so $\pi^* \omega_t$ is "regular";

The normalized Kähler-Ricci flow

The normalized Kähler-Ricci flow on X can be written as

$$\frac{(\omega_t + dd^c \varphi_t)^n}{C_n^\kappa e^{-(n-\kappa)t}} = e^{\dot{\varphi}_t + \varphi_t} v(h_A).$$

Particular case of (CMAF) considered above with $T = +\infty$, $h \equiv 0$ and

- $e^{\psi} dV = \pi^* v(h_A)$ if $\pi : \tilde{X} \to X$ =desingularization of X;
- $\omega_t = e^{-t}\omega_0 + (1 - e^{-t}) f^* \omega_A$ so $\pi^* \omega_t$ is "regular";
- normalization s.t. the volume of LHS cv to 1 as $t \to +\infty$.

The normalized Kähler-Ricci flow

The normalized Kähler-Ricci flow on X can be written as

$$\frac{(\omega_t + dd^c\varphi_t)^n}{C_n^\kappa e^{-(n-\kappa)t}} = e^{\dot{\varphi}_t + \varphi_t} v(h_A).$$

Particular case of (CMAF) considered above with $T = +\infty$, $h \equiv 0$ and

- $e^\psi dV = \pi^* v(h_A)$ if $\pi : \tilde{X} \to X$ =desingularization of X;
- $\omega_t = e^{-t}\omega_0 + (1 - e^{-t}) f^*\omega_A$ so $\pi^*\omega_t$ is "regular";
- normalization s.t. the volume of LHS cv to 1 as $t \to +\infty$.

Theorem (EGZ16)

If ρ smooth on a Zariski open set then φ_t cv to $\varphi_\infty := \varphi_{can} \circ f$.

The normalized Kähler-Ricci flow

The normalized Kähler-Ricci flow on X can be written as

$$\frac{(\omega_t + dd^c \varphi_t)^n}{C_n^\kappa e^{-(n-\kappa)t}} = e^{\dot{\varphi}_t + \varphi_t} v(h_A).$$

Particular case of (CMAF) considered above with $T = +\infty$, $h \equiv 0$ and

- $e^\psi dV = \pi^* v(h_A)$ if $\pi : \tilde{X} \to X$ =desingularization of X;
- $\omega_t = e^{-t}\omega_0 + (1 - e^{-t}) f^* \omega_A$ so $\pi^*\omega_t$ is "regular";
- normalization s.t. the volume of LHS cv to 1 as $t \to +\infty$.

Theorem (EGZ16)

If ρ smooth on a Zariski open set then φ_t cv to $\varphi_\infty := \varphi_{can} \circ f$.

Result due to Song-Tian [ST12] when X is smooth.

Idea of the proof

- Construct a subsolution $u(t,x)$ of the flow such that

$$\varphi_\infty \leq \lim_{t \to +\infty} u(t,x).$$

Idea of the proof

- Construct a subsolution $u(t,x)$ of the flow such that

$$\varphi_\infty \leq \lim_{t \to +\infty} u(t,x).$$

- Would like to construct a supersolution $v(t,x)$ such that

$$\varphi_\infty \geq \lim_{t \to +\infty} v(t,x).$$

Idea of the proof

- Construct a subsolution $u(t,x)$ of the flow such that

$$\varphi_\infty \leq \lim_{t \to +\infty} u(t,x).$$

- Would like to construct a supersolution $v(t,x)$ such that

$$\varphi_\infty \geq \lim_{t \to +\infty} v(t,x).$$

- The comparison principle insures $\varphi_\infty = \lim_{t \to +\infty} \varphi(t,x)$.

Idea of the proof

- Construct a subsolution $u(t,x)$ of the flow such that

$$\varphi_\infty \leq \lim_{t \to +\infty} u(t,x).$$

- Would like to construct a supersolution $v(t,x)$ such that

$$\varphi_\infty \geq \lim_{t \to +\infty} v(t,x).$$

- The comparison principle insures $\varphi_\infty = \lim_{t \to +\infty} \varphi(t,x)$.
- OK for subsolution if ρ is ω_0-psh; extra work otherwise.

Idea of the proof

- Construct a subsolution $u(t,x)$ of the flow such that

$$\varphi_\infty \leq \lim_{t \to +\infty} u(t,x).$$

- Would like to construct a supersolution $v(t,x)$ such that

$$\varphi_\infty \geq \lim_{t \to +\infty} v(t,x).$$

- The comparison principle insures $\varphi_\infty = \lim_{t \to +\infty} \varphi(t,x)$.
- OK for subsolution if ρ is ω_0-psh; extra work otherwise.
- More involved to provide an accurate supersolution.

Constructing rough sub/supersolutions

Easy uniform bound from above : $v(t,x) :\equiv C >> 1$ is a supersolution.

Constructing rough sub/supersolutions

Easy uniform bound from above : $v(t,x) :\equiv C >> 1$ is a supersolution.

- Indeed $v(0,x) = C \geq \varphi_0$ if $C \geq \sup_X \varphi_0$ and

Constructing rough sub/supersolutions

Easy uniform bound from above : $v(t,x) :\equiv C >> 1$ is a supersolution.

- Indeed $v(0,x) = C \geq \varphi_0$ if $C \geq \sup_X \varphi_0$ and
- $(\omega_t + dd^c v)^n = \sum_{j=0}^{\kappa} C_n^j e^{-(n-j)t}(1-e^{-t})^j f^*\omega_A^j \wedge \omega_0^{n-j}$

Constructing rough sub/supersolutions

Easy uniform bound from above : $v(t,x) :\equiv C >> 1$ is a supersolution.

- Indeed $v(0,x) = C \geq \varphi_0$ if $C \geq \sup_X \varphi_0$ and
- $(\omega_t + dd^c v)^n = \sum_{j=0}^{\kappa} C_n^j e^{-(n-j)t}(1-e^{-t})^j f^*\omega_A^j \wedge \omega_0^{n-j}$
  $\leq C_n^{\kappa} e^{-(n-\kappa)t} e^B v(h_A) \leq C_n^{\kappa} e^{-(n-\kappa)t} e^{\dot{v}_t + v_t} v(h_A)$

Constructing rough sub/supersolutions

Easy uniform bound from above : $v(t,x) :\equiv C >> 1$ is a supersolution.

- Indeed $v(0,x) = C \geq \varphi_0$ if $C \geq \sup_X \varphi_0$ and
- $(\omega_t + dd^c v)^n = \sum_{j=0}^{\kappa} C_n^j e^{-(n-j)t}(1-e^{-t})^j f^*\omega_A^j \wedge \omega_0^{n-j}$
$$\leq C_n^{\kappa} e^{-(n-\kappa)t} e^B v(h_A) \leq C_n^{\kappa} e^{-(n-\kappa)t} e^{\dot{v}_t + v_t} v(h_A)$$
if $C \geq B$.

Constructing rough sub/supersolutions

Easy uniform bound from above : $v(t,x) :\equiv C >> 1$ is a supersolution.

- Indeed $v(0,x) = C \geq \varphi_0$ if $C \geq \sup_X \varphi_0$ and
- $(\omega_t + dd^c v)^n = \sum_{j=0}^{\kappa} C_n^j e^{-(n-j)t}(1-e^{-t})^j f^*\omega_A^j \wedge \omega_0^{n-j}$
  $\leq C_n^{\kappa} e^{-(n-\kappa)t} e^B v(h_A) \leq C_n^{\kappa} e^{-(n-\kappa)t} e^{\dot{v}_t + v_t} v(h_A)$
  if $C \geq B$.

Showing a uniform bound from below is more involved and relies on

Constructing rough sub/supersolutions

Easy uniform bound from above : $v(t,x) :\equiv C >> 1$ is a supersolution.

- Indeed $v(0,x) = C \geq \varphi_0$ if $C \geq \sup_X \varphi_0$ and
- $(\omega_t + dd^c v)^n = \sum_{j=0}^{\kappa} C_n^j e^{-(n-j)t}(1-e^{-t})^j f^*\omega_A^j \wedge \omega_0^{n-j}$
  $\leq C_n^\kappa e^{-(n-\kappa)t} e^B v(h_A) \leq C_n^\kappa e^{-(n-\kappa)t} e^{\dot{v}_t + v_t} v(h_A)$
  if $C \geq B$.

Showing a uniform bound from below is more involved and relies on

Theorem (Kolodziej98,EGZ08, Demailly-Pali10)

Assume

$$V_t^{-1}(\omega_t + dd^c\psi_t)^n = F_t dV_X$$

with F_t uniformly in $L^{1+\varepsilon}$ then ψ_t is uniformly bounded.

Construction of a fine subsolution

Consider

$$u(t,x) = (1 - e^{-t})\varphi_\infty(x) + e^{-t}\rho(x) - Ce^{-t} + h(t)$$

Construction of a fine subsolution

Consider

$$u(t,x) = (1-e^{-t})\varphi_\infty(x) + e^{-t}\rho(x) - Ce^{-t} + h(t)$$

where $h + h' = \kappa \log(1-e^{-t})$, $h(0) = 0$. Then

Construction of a fine subsolution

Consider

$$u(t,x) = (1-e^{-t})\varphi_\infty(x) + e^{-t}\rho(x) - Ce^{-t} + h(t)$$

where $h + h' = \kappa \log(1-e^{-t})$, $h(0) = 0$. Then

- $x \mapsto u(t,x)$ is ω_t-psh if ρ is ω_0-psh

Construction of a fine subsolution

Consider

$$u(t,x) = (1-e^{-t})\varphi_\infty(x) + e^{-t}\rho(x) - Ce^{-t} + h(t)$$

where $h + h' = \kappa \log(1-e^{-t})$, $h(0) = 0$. Then

- $x \mapsto u(t,x)$ is ω_t-psh if ρ is ω_0-psh
- $u_0(x) = \rho(x) - C \leq \varphi_0$ if $C >> 1$

Construction of a fine subsolution

Consider

$$u(t,x) = (1-e^{-t})\varphi_\infty(x) + e^{-t}\rho(x) - Ce^{-t} + h(t)$$

where $h + h' = \kappa \log(1-e^{-t})$, $h(0) = 0$. Then

- $x \mapsto u(t,x)$ is ω_t-psh if ρ is ω_0-psh
- $u_0(x) = \rho(x) - C \leq \varphi_0$ if $C >> 1$
- $\dot{u}_t + u_t = \varphi_\infty + h' + h$ hence

Construction of a fine subsolution

Consider

$$u(t,x) = (1-e^{-t})\varphi_\infty(x) + e^{-t}\rho(x) - Ce^{-t} + h(t)$$

where $h + h' = \kappa \log(1-e^{-t})$, $h(0) = 0$. Then

- $x \mapsto u(t,x)$ is ω_t-psh if ρ is ω_0-psh
- $u_0(x) = \rho(x) - C \leq \varphi_0$ if $C >> 1$
- $\dot{u}_t + u_t = \varphi_\infty + h' + h$ hence

$$(\omega_t + dd^c u_t)^n \geq C_n^\kappa e^{-(n-\kappa)t}(1-e^{-t})^\kappa T_{can}^\kappa \wedge \omega_{SF}^{n-\kappa}$$

Construction of a subsolution

Consider

$$u(t,x) = (1-e^{-t})\varphi_\infty(x) + e^{-t}\rho(x) - Ce^{-t} + h(t)$$

where $h + h' = \kappa \log(1-e^{-t})$, $h(0) = 0$. Then

- $x \mapsto u(t,x)$ is ω_t-psh if ρ is ω_0-psh
- $u_0(x) = \rho(x) - C \leq \varphi_0$ if $C >> 1$
- $\dot{u}_t + u_t = \varphi_\infty + h' + h$ hence

$$(\omega_t + dd^c u_t)^n \geq C_n^\kappa e^{-(n-\kappa)t}(1-e^{-t})^\kappa T_{can}^\kappa \wedge \omega_{SF}^{n-\kappa} = C_n^\kappa e^{-(n-\kappa)t} e^{\dot{u}_t + u_t} v(h_A)$$

Construction of a subsolution

Consider

$$u(t,x) = (1-e^{-t})\varphi_\infty(x) + e^{-t}\rho(x) - Ce^{-t} + h(t)$$

where $h + h' = \kappa \log(1-e^{-t})$, $h(0) = 0$. Then

- $x \mapsto u(t,x)$ is ω_t-psh if ρ is ω_0-psh
- $u_0(x) = \rho(x) - C \leq \varphi_0$ if $C >> 1$
- $\dot{u}_t + u_t = \varphi_\infty + h' + h$ hence

$$(\omega_t + dd^c u_t)^n \geq C_n^\kappa e^{-(n-\kappa)t}(1-e^{-t})^\kappa T_{can}^\kappa \wedge \omega_{SF}^{n-\kappa} = C_n^\kappa e^{-(n-\kappa)t} e^{\dot{u}_t + u_t} v(h_A)$$

Thus u subsolution and lower bound with exp. speed:

Construction of a subsolution

Consider

$$u(t,x) = (1-e^{-t})\varphi_\infty(x) + e^{-t}\rho(x) - Ce^{-t} + h(t)$$

where $h + h' = \kappa \log(1-e^{-t})$, $h(0) = 0$. Then

- $x \mapsto u(t,x)$ is ω_t-psh if ρ is ω_0-psh
- $u_0(x) = \rho(x) - C \leq \varphi_0$ if $C >> 1$
- $\dot{u}_t + u_t = \varphi_\infty + h' + h$ hence

$$(\omega_t + dd^c u_t)^n \geq C_n^\kappa e^{-(n-\kappa)t}(1-e^{-t})^\kappa T_{can}^\kappa \wedge \omega_{SF}^{n-\kappa} = C_n^\kappa e^{-(n-\kappa)t} e^{\dot{u}_t + u_t} v(h_A)$$

Thus u subsolution and lower bound with exp. speed:

$$-C'(t+1)e^{-t} + e^{-t}\rho(x) \leq (\varphi_t - \varphi_\infty)(x).$$

Constructing a fine supersolution

The construction of an efficient supersolution is harder :

Constructing a fine supersolution

The construction of an efficient supersolution is harder :

- Need to control mixed terms $(f^*\omega_A + dd^c\varphi_\infty)^j \wedge \omega_{SF}^{n-j}$

Constructing a fine supersolution

The construction of an efficient supersolution is harder :

- Need to control mixed terms $(f^*\omega_A + dd^c\varphi_\infty)^j \wedge \omega_{SF}^{n-j}$
- Use that φ_∞ and ρ are smooth in $X_{can} \setminus D$;

Constructing a fine supersolution

The construction of an efficient supersolution is harder :

- Need to control mixed terms $(f^*\omega_A + dd^c\varphi_\infty)^j \wedge \omega_{SF}^{n-j}$
- Use that φ_∞ and ρ are smooth in $X_{can} \setminus D$;
- Establish a comparison principle for Kähler manifolds with boundary

Constructing a fine supersolution

The construction of an efficient supersolution is harder :

- Need to control mixed terms $(f^*\omega_A + dd^c\varphi_\infty)^j \wedge \omega_{SF}^{n-j}$
- Use that φ_∞ and ρ are smooth in $X_{can} \setminus D$;
- Establish a comparison principle for Kähler manifolds with boundary
- Construct a supersolution in $X \setminus V_\varepsilon(D)$ and conclude.

Constructing a fine supersolution

The construction of an efficient supersolution is harder :

- Need to control mixed terms $(f^*\omega_A + dd^c\varphi_\infty)^j \wedge \omega_{SF}^{n-j}$
- Use that φ_∞ and ρ are smooth in $X_{can} \setminus D$;
- Establish a comparison principle for Kähler manifolds with boundary
- Construct a supersolution in $X \setminus V_\varepsilon(D)$ and conclude.

More technical details in our paper.

The end

- [ST12] J.Song, G.Tian: Canonical measures and Kähler-Ricci flow. J. Amer. Math. Soc. **25** (2012), no. 2, 303-353.
- [Choi15] Y.-J.Choi: Semi-positivity of fiberwise Ricci-flat metrics on Calabi-Yau fibrations. Preprint arXiv:1508.00323
- [EGZ14] Weak solutions to degenerate complex Monge-Ampère flows II. Preprint arXiv:1407.2504
- [EGZ16] Convergence of weak Kähler-Ricci Flows on minimal models of positive Kodaira dimension. Preprint arXiv:1604.07001.