Convergence of Ricci Flows with Bounded Scalar Curvature

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We will start with an elementary problem in Ricci flow. Consider a smooth family of metrics $g(t)$ with $t \in [0,T)$ on a compact manifold M^n. The Ricci flow equation reads $\partial_t g(t) = -2Ric_{g(t)}$ and we are guaranteed solutions to this equation on some maximal time interval $[0,T)$ after which we may reach a singularity. Assume $T < \infty$. The result from Hamilton '82 is that at a maximal time T we have $\max_M |Rm|(\cdot,t) \to \infty$ as $t \to T$. A more refined result by Sesum '03 shows that the norm of the Ricci tensor $|Ric|$ becomes unbounded on $M \times [0,T)$. This yields a natural question/conjecture?

Conjecture 1 *Does the scalar curvature R become unbounded on $M \times [0,T)$?*

A remark is that this conjecture is true when $n \in \{2,3\}$ where $n = 3$ is proved by the Hamilton-Ivey pinching, and the result is also true for Kähler manifolds. We also have

$$\partial_t R \;=\; \Delta R + 2|Ric|^2 \;\geq\; \Delta R$$

hence by the maximum principle we have that R is bounded from below, so $R > -C$. It'd be interesting as well to consider the contrapositive of this conjecture, or rather

1. Assuming that $R < C$ on $M \times [0,T)$, what is the behavior of the metric $g(t)$ as $t \to T$?
2. Assume that $R(\cdot,t) < \frac{C}{T-t}$ like in a type 1 singularity, and that $diam_t M < C\sqrt{T-t}$, what is the behavior of the rescaled metric $\frac{g(t)}{T-t}$ as $t \to T$?

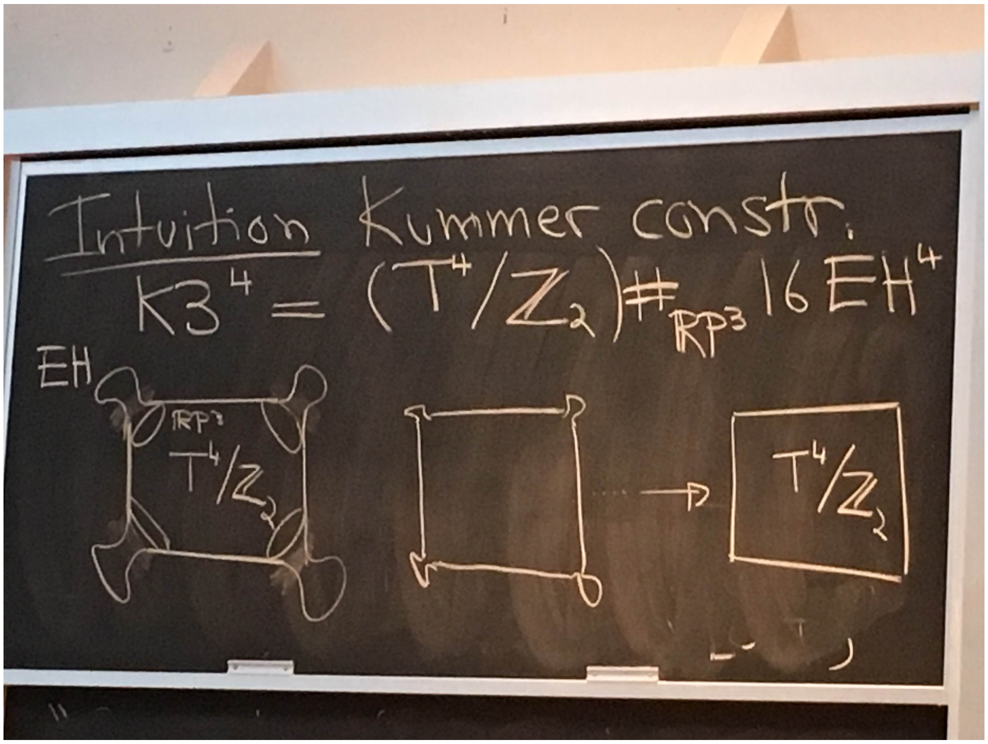
Question 2 here is interesting here since Perelman proved that if M is a Fano manifold an g is a Kähler Ricci flow, then both the inequalities listed are true. This yields another conjecture

Conjecture 2 (Hamilton-Tian) *If $g(t)$ is a Kähler-Ricci flow and M is Fano, then $(T-t)^{-1}g(t)$ subconverges to a gradient shrinking soliton away from codimension ≥ 4 as $t \to T$.*

We will discuss for now the case that $n = 4$. Last year there was a theorem proved in this direction:

Theorem 1 (Bamler-Zhang '15, Simon '15) *If $n = 4$ and $R < C$ on $M \times [0,T)$, then*

1. *$\int_M |Rm|^2(\cdot,t) < C$ for all $t \in [0,T)$,*
2. *$\int_M |Ric|^{4-\epsilon}(\cdot,t) < C_\epsilon$ for all $t \in [0,T)$ and $\epsilon > 0$,*
3. *$\int_0^T \int_M |Ric|^4 < \infty$,*
4. *$(M,g(t))$ in the limit as $t \to T$ approaches a C^0 orbifold $(M_T, g(T))$, and*
5. *the Ricci flow can be continued from $(M_T, g(T))$, hence we obtain a smooth orbifold structure.*



We can gain some intuition behind this theorem by picking an example. Recall the Kummer construction where we have a surface

$$K3^4 \;=\; (\mathbb{T}^4/\mathbb{Z}_2)\;\#_{\mathbb{RP}^3}\;16EH^4.$$

Cross-sections around the corners of $\mathbb{T}^4/\mathbb{Z}_2$ are diffeomorphic to $\mathbb{RP}^3$, so we can glue in Eguchi-Hanson metrics along these copies of $\mathbb{RP}^3$. We therefore obtain an almost Ricci flat metric on $K3^4$. In this gluing process, we can choose the size of the the Eguchi-Hanson metrics arbitrarily small. So our gluing process generates a family of almost Ricci flat metrics that degenerate to the original orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

A third question naturall arises: **Can such a degeneration occur in a Ricci flow?** In higher dimensions we will look at a Ricci flow $g(t)$ on $M \times [0,T)$ where $T < \infty$.

We now move to higher dimensions $n \geq 4$. In the setting of Question 1, we know the following:

Theorem 2 (Bamler-Zhang '15) *If we have an upper bound C on our scalar curvature $R < C$ on $M \times [0,T)$ then $d_T = \lim_{t\to T} d_t$ exists and is a pseudometric.*

Theorem 3 (Bamler '15) *If $R < C$ on $M \times [0,T)$ then there exists an open subset $\mathcal{R} \subset M$ such that*

1. *$g(t) \to g(T)$ in a C^∞ manner as $t \to T$, when restricted to $\mathcal{R}$.*
2. *given the relation $x \sim y$ if and only if $d(x,y) = 0$, we have that*

$$(M_T := M/\sim, d_T) \;\cong\; \overline{(\mathcal{R}, g(T))}$$

   where the overbar indicates the completion of the submanifold $\mathcal{R}$.

3. *$\dim_{\mathcal{M}}(M_T \setminus \mathcal{R}) \leq n-4$, where $\dim_{\mathcal{M}}$ denotes the Minkowski dimension.*

We have an additional theorem in the setting of Question 2;

Theorem 4 (Bamler '15) *If $R(\cdot,t) < \frac{C}{T-t}$ and $diam_t M < C\sqrt{T-t}$ then for any $t_i \to T$ there is a subsequence such that*

$$\left(M, \frac{g(t_i)}{T-t_i}\right) \longrightarrow_{i\to\infty} (X, d_X)$$

and $X = \mathcal{R} \cup \mathcal{S}$ such that

1. *$d_X|_{\mathcal{R}}$ is isometric to the length metric of a smooth Riemannian manifold g_∞.*
2. *$\frac{g(t_i)}{T-t_i} \to g_\infty$ on $\mathcal{R}$ in a C^∞ manner as $i \to \infty$.*
3. *There exists $f \in C^\infty(\mathcal{R})$ such that $Ric_{g_\infty} + \nabla^2 f = \frac{1}{2} g_\infty$ on $\mathcal{R}$.*
4. *$\dim_{\mathcal{M},d_X}(\mathcal{S}) \leq n-4$.*

There is a corollary to this theorem, which is

Corollary 1 *The Hamilton-Tian conjecture is true.*

A remark about this is that there is progress in the complex geometric setting currently being addressed by Tian-Zhang and Chen-Wang. Further results include a compactness theorem (Bamler '15). Particularly, if $(M_i \times [-2,0], g_i(t))$ are Ricci flows with $R_{g_i(t)} < C$ on $M_i \times [-2,0]$, and $\mu[g_i(-2), \frac{1}{2}] > -C$. We also have that there exists a subsequence so that

$$(M_i, g_i(0)) \to (X, d_X, \mathcal{R}, g_\infty)$$

just as before with $\dim_{\mathcal{M}}(X \setminus \mathcal{R}) \leq n-4$.

Definition 1 (Curvature Radius) *Let (M,g) be a Riemannian manifold. We define the **curvature radius** to be the quantity*

$$r_{Rm}(x) := \sup\{r > 0 : |Rm| < \frac{1}{r^2} \text{ on } B(x,r)\}.$$

There is additionally a curvature bound result (Bamler '15): Let $(M \times [-2,0], g(t))$ be a Ricci flow with

1. $R < A$ on $M \times [-2,0]$, and
2. $\mu\left(g(-2), \frac{1}{2}\right) > -A$.

then with $0 < r < 1$ we have

$$\int_{B(x,0,r)} |Rm|^{2-\epsilon}(\cdot,0) \leq \int_{B(x,0,r)} r_{Rm}^{-4+2\epsilon}(\cdot,0) < C(A,\epsilon) r^{n-4+2\epsilon}.$$

There are some ingredients that are used in setting up this proof: Suppose for the remaining discussion that $R < 1$ on $M \times [-2,0]$, and that we have $\mu\left(g(-2), \frac{1}{2}\right) > -A$, where A is a lower entropy bound.

A result by Perelman, Zhang, and Chen-Wang shows that for all $(x,t) \in M \times [-1,0]$ with $0 < r < 1$ we have that

$$k_1(A) r^n < |B(x,t,r)|_t < k_2(A) r^n.$$

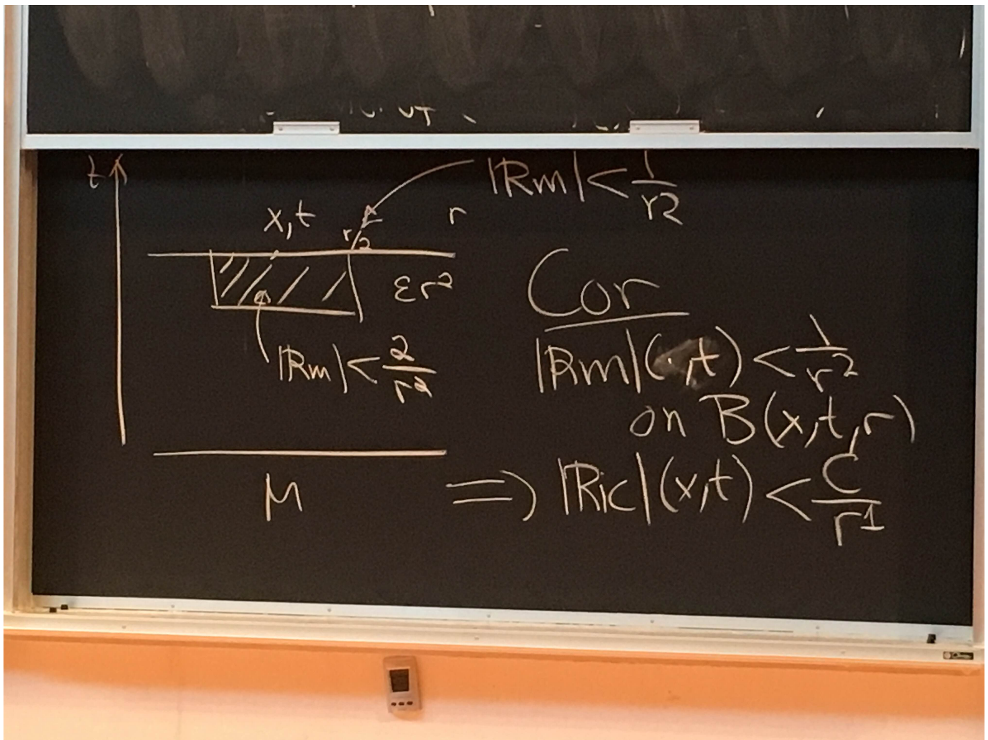
A result by Bamler-Zhang shows that if $t_1, t_2 \in [-2,0]$ and $d_{t_1}(x,y) < D$ then

$$|d_{t_1}(x,y) - d_{t_2}(x,y)| < C(D)\sqrt{|t_1 - t_2|}.$$

Another result by Bamler-Zhang is that if $-1 \leq s < t \leq 0$ with $x, y \in M$ we have lower and upper Gaussian bounds:

$$\frac{1}{C(t-s)^{n/2}}\exp\left(\frac{Cd_s^2(x,y)}{t-s}\right) \;<\; k(x,t,y,s) \;<\; \frac{C}{(t-s)^{n/2}}\exp\left(\frac{d_s^2(x,y)}{C(t-s)}\right).$$

where k is a heat kernel of the heat equation $\partial_t - \Delta = 0$ coupled with Ricci flow. Lastly there is a backwards pseudo-locality bound (Bamler-Zhang) where if $0 < r < 1$ and $(x,t) \in M \times [-1,0]$ with the conditions that $|Rm|(\cdot,t) < \frac{1}{r^2}$ on $B(x,t,r)$ then this implies that $|Rm| < \frac{2}{r^2}$ on $B(x,t,\frac{r}{2}) \times [t - \epsilon r^2, t]$.



This yields a corollary:

Corollary 2 *If* $|Rm|(\cdot,t) < \frac{1}{r^2}$ *on* $B(x,t,r)$ *then* $|Ric|(x,t) < \frac{C}{r^1}$.

We also provide a quick idea of the proof of the compactness theorem mentioned earlier. Suppose a priori that

$$\int_{B(x,t,r)} r_{RM}^{-3.1}(\cdot,t) \;<\; Er^{n-3.1}$$

We can deduce that under this a priori assumption, every blowup of a Ricci flow has the form $(X, d_X, \mathcal{R}, g)$, as in the previously mentioned compactness theorem. Moreover, this blowup is Ricci flat on its regular subset. Using the results of Cheeger-Colding-Naber, it is then possible to deduce L^p bounds on r_{RM}^{-1}, which improve the a priori assumptions.