Ancient Solutions, Their Asymptotics, and Uniqueness

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We will survey some results done with ancient and eternal solutions of geometric flows. They occur as blow-up limits around singularities. Heuristically, ancient and eternal solutions are those that are defined on $(-\infty, T)$ with $T \leq +\infty$. We will discuss blow up limits around singularities, classification results, and methods of constructing new ancient solutions, as well as the Ricci flow, mean curvature flow.

Definition 1 *A solution $u(\cdot, t)$ is called **ancient** if it exists for $t \in (-\infty, T)$ for $T < \infty$, and they occur as blow up limits of Type I singularities. We say that $u(\cdot, t)$ is eternal for all $t \in (-\infty, \infty)$ and they occur as blow up limits of Type II singularities.*

Some examples of these include

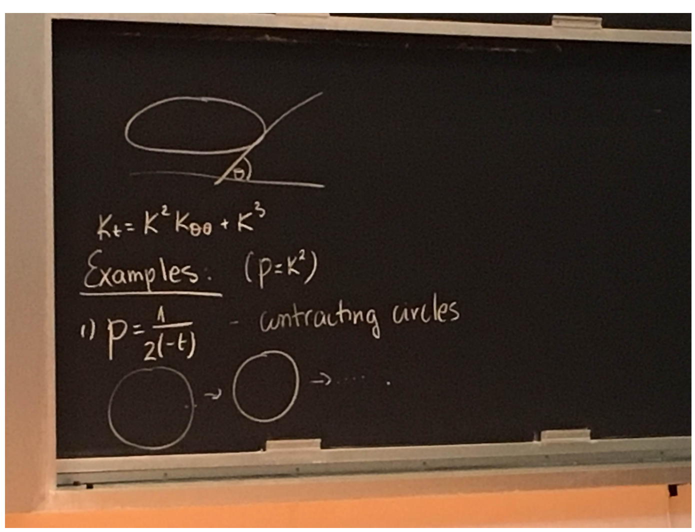
- Solitons (sphere, cylinders, translating solitons).
- A result of Hamilton shows the derivation of the differentiable Harnack inequalities for the Ricci and mean curvature flows, where $Rm > 0$ for the Ricci flow, and $A > 0$ for MCF. He used them to show that all eternal solutions with $Rm > 0$ (in RF), and $A > 0$ (in MCF) that attain the spacetime maximum of R (in RF) and H (in MCF) must be steady solitons (RF), or translating solitons (MCF).

Definition 2 *A **Type I ancient solution** for Ricci flows are those that satisfy*

$$\limsup_{t \to -\infty} |t| \sup_{M_t} |Rm| \ < \ \infty.$$

*Otherwise they are **Type II** solutions.*

For mean curvature flow, suppose we have an immersion $F : M^n \hookrightarrow \mathbb{R}^{n+1}$ with $\frac{\partial F}{\partial t} = -H\nu$ subject to $F(\cdot, 0) = F_0$. For $n = 1$ we have that F_t is an embedded, closed, convex ancient solution to the CSF, in which case we have that $\frac{\partial}{\partial t} X = -\kappa\nu$ which is equivalent to $\frac{\partial}{\partial t}\kappa = \kappa_{ss} + \kappa^3$. Some examples of the 1-dimensional cases include contracti



1. $p = \frac{1}{2(-t)}$ which correspond to contracting circles,

2. Angenant ovals where for $\lambda > 0$ we have

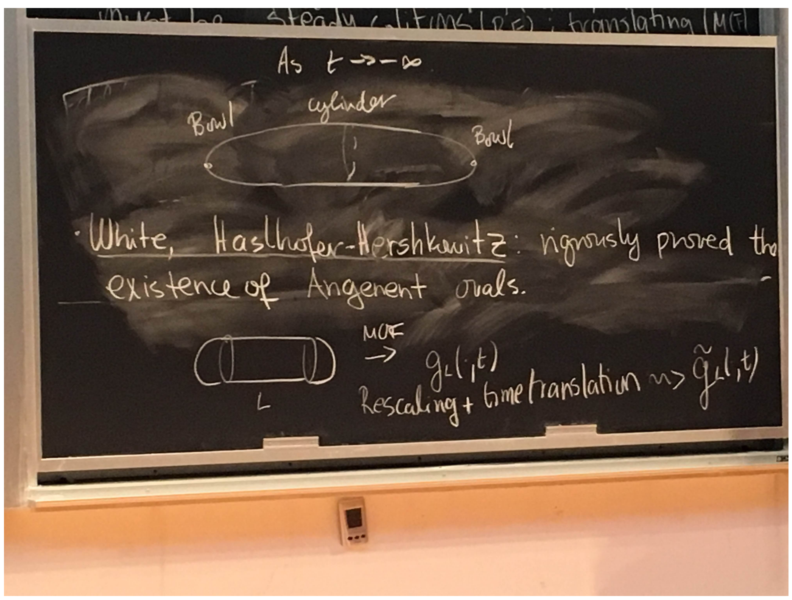
$$p(\theta, t) \;=\; \lambda\left(\frac{1}{1 - e^{-2\lambda t}} - \sin^2(\theta + r)\right)$$

   which correspond to grim reaper flows. When we take the limit as $t \to -\infty$ we will see two parallel lines.

Theorem 1 (Daskalopolous-Hamilton-Sesum) *The only closed ancient embedded convex solutions to the CSF are either circles or Angenent ovals.*

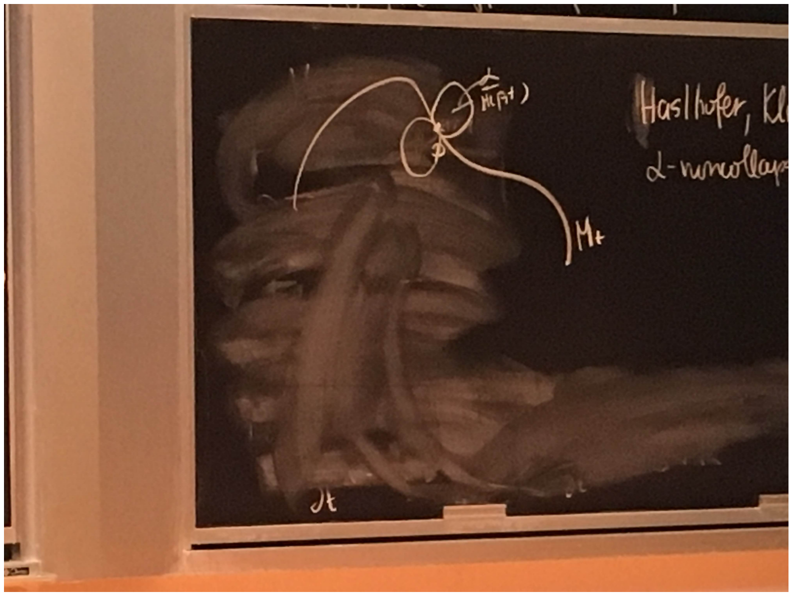
It's natural now to ask what happens in higher dimensions. What about $n \geq 2$.

Conjecture 1 (Angenent) *There exist ancient, convex, closed solutions to the mean curvature flow that as $t \to -\infty$ which after rescaling look more and more like cylinders and with a blow down limit at the tips look like translating solitons (bowls).*



Papers by White, and Haslhofer-Hershkowitz rigorously proves the existence of Angenent ovals. Our goal is to prove that the only ancient, noncollapsed, closed solutions to the mean curvature flow are either the spheres or Angenent solutions. We define noncollapsing here:

Definition 3 *Let $\alpha > 0$. A mean convex mean curvature flow $\{M_t\}_{t>0}$ is called α-**noncollapsed** if there exist for every p exterior and interior balls of radius at least $\frac{\alpha}{H(p)}$ tangent to the hypersurface at p.*



A nice result by Haslhofer-Kleiner shows that if you have α-noncollapsed and ancient solutions, you also have convexity. There are more results from White and Haslhofer-Hershkowitz. Additionally we have

$$\lim_{t\to-\infty} \frac{1}{\sqrt{|t|}}\ diam(M_t) \quad = \quad \infty$$

and

$$\limsup_{t\to-\infty} \sqrt{|t|}\ \ \sup_{M_t} |A| \quad = \quad \infty$$

both of which hold for the Angenent solution and the second holds for Type II solutions. Results by Huisken-Sinestrari, and Haslhofer-Hershkowitz follows as such: Let $\{M_t\}_{t>0}$ be an ancient, noncollapsed, closed, mean curvature flow. Then if one of the following holds:

1. $\limsup_{t\to-\infty} |t|^{-1/2} diam(M_t) < \infty$, or
2. $\limsup_{t\to-\infty} |t|^{1/2} \sup_{M_t} |A| < \infty$ for Type I solutions

then M_t is just a family of contracting spheres. From now on we will refer to **ancient ovals** as closed, noncollapsed, and ancient solutions to the mean curvature flow. We will consider ancient ovals that are $O(1)\times O(n)$ invariant hypersurfaces in $\mathbb{R}^{n+1}$ where $|y| = u(x,t)$ for $y \in \mathbb{R}^n$ and $x \in \mathbb{R}$ that satisfy the equation

$$\frac{\partial}{\partial t} u \quad = \quad \frac{u_{xx}}{1+u_x^2} - \frac{1}{u}.$$

By using the rescaling $\tau = -\log(-t)$ we get that

$$\frac{\partial}{\partial t} u \quad = \quad \frac{u_{yy}}{1+u_y^2} - \frac{y}{2} u_y - \frac{n-1}{u} + \frac{u}{2}$$

whose solutions are $u(y,\tau) = \sqrt{2(n-1)}$-cylinders. Work by X. J. Wang show that dialations

$$\{\widetilde{X} \in \mathbb{R}^{n+1} \mid (-t)^{1/2} \widetilde{X} \in M_t\}$$

that sweep out the whole space converge as $t \to -\infty$ to either

1. a sphere, or

2. a cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ of radius $\sqrt{2(n-1)}$.

Theorem 2 (Angenent, Daskalopolous, Sesum) *Let $\{M_t\}$ be an ancient oval as above. Then either $\{M_t\}$ is a family of contracting spheres, or $u(y,\tau)$ has the following asymptotics:*

1. ***Parabolic region**: $u(y,\tau) = \sqrt{2(n-1)}\left(1 + \frac{y^2-2}{4\tau}\right) + O(|\tau|^{-1})$ with $|y| \le M$,*
2. ***Intermediate region**: $z = \frac{y}{\sqrt{|\tau|}}$ where $\overline{u}(z,\tau) = u(z\sqrt{|\tau|},\tau)$ and $u(z,\tau) \to \sqrt{2-z^2}$ as $\tau \to -\infty$, or*
3. ***Tip region**: There's a rescaling which results in a translating soliton.*

Corollary 1 *As a result of the asymptotics, we have that*

$$\begin{aligned} |t|^{-1/2} diam(M_t) &\approx \sqrt{8\log(t)} \\ \sqrt{|t|} H_{\max}(M_t) &\approx \frac{\sqrt{\log(t)}}{2} \end{aligned}$$

as $t \to -\infty$.

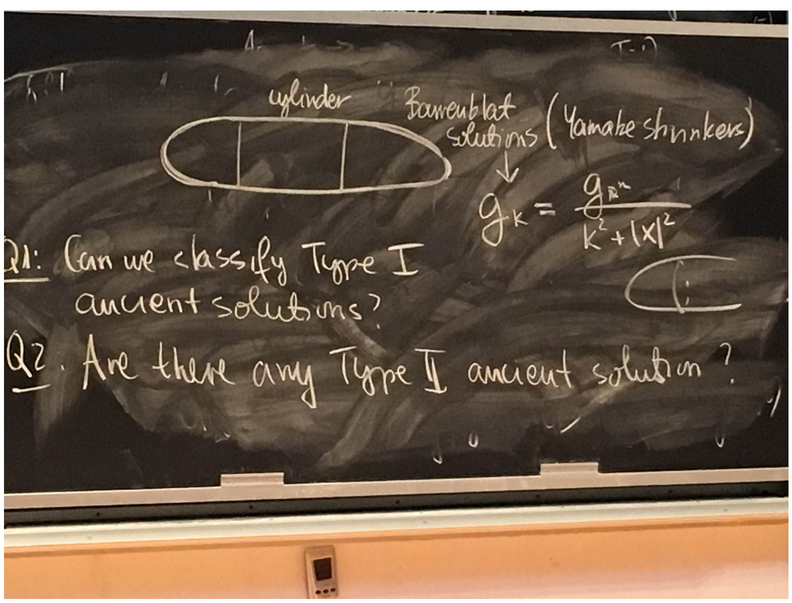
Conjecture 2 *The ancient ovals with $O(1) \times O(n)$ symmetry are uniquely determined by their asymptotics.*

Recall the $\frac{\partial}{\partial t} g_{ij} = -R g_{ij}$. For $n = 2$ we have that the ancient flow is just the Ricci flow. We consider closed, ancient solutions on $\mathbb{S}^n$. For $n \ge 3$ Yamabe flow is conformal so we have $g = v^{\frac{4}{n-2}} g_{\mathbb{S}^n}$ and we have that

$$\frac{\partial}{\partial t} v^{\frac{n+2}{n-2}} = \Delta_{\mathbb{S}^n} v - c(n) v.$$

We now list some examples of ancient solutions:

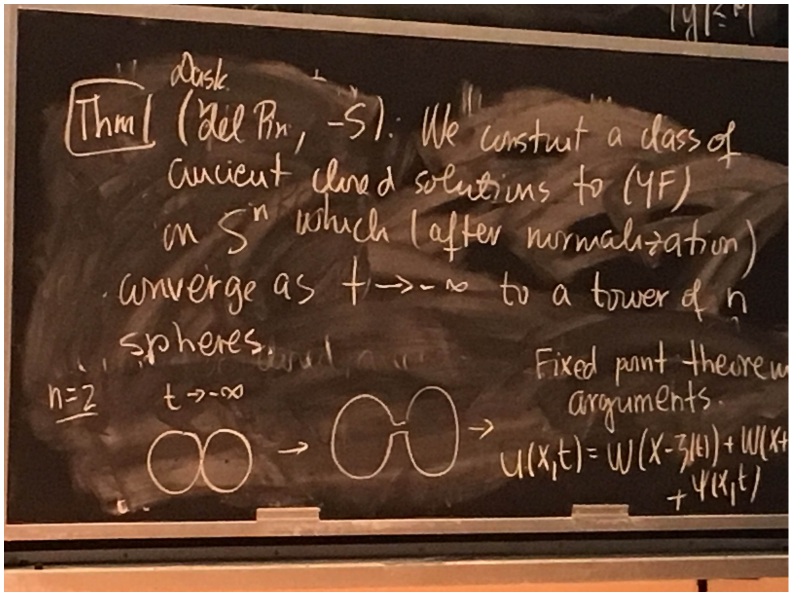
1. Spheres,
2. Work by King-Brendle show that $g = \frac{g_{\mathbb{R}^n}}{a(t)+2b(t)|x|^2+a(t)|x|^4}$. An example of this are the Barenblat solutions of Yamabe shrinkers $g_k = \frac{g_{\mathbb{R}^n}}{k^2+|x|^2}$.



Both of the following are Type I solutions, so two questions naturally arise. Can we classify Type I solutions? And, are there any Type II ancient solutions?

Theorem 3 (Daskalopolous, del-Pino, Sesum) *We construct a class of ancient closed solutions to Yamabe flow on $\mathbb{S}^n$ which (after normalization) converge as $t \to -\infty$ to a tower of n spheres.*

For $n = 2$ we have the following image of the ancient solutions:



This is proved roughly by a fixed-point theorem for arguments of the form $u(x,t) = w(x - \xi(t)) + w(x + \xi(t)) + \psi(x,t)$. We have that for every $\lambda \geq 1$ where $\lambda = 1$ corresponds to the Barenblat solution, there exists a Yamabe shrinker (a traveling wave of speed λ) where $v_\lambda(x,t) = V(x - \lambda\tau) = 1 - C_\lambda e^{-\gamma_\lambda y} + \mathcal{O}(e^{-\gamma_\lambda y})$ as $y \to \infty$. A theorem by Daskalopolous, del-Pino, King, Sesum details the wave properties of these asymptotics.