The Chern-Ricci Flow

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The Chern-Ricci flow is a parabolic flow on compact manifolds, and we will discuss a long-time existence of these flows, as well as their behavior on complex surfaces which are minimal (to be discussed), as well as non-minimal.

Let M be a compact, complex manifold with $\dim_{\mathbb{C}} M = n$. Let g_o be a Hermitian metric which we locally write $(g_o)_{i\bar{j}}$, associated with the (1,1) form

$$\omega_o = \sqrt{-1}(g_o)_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

where ω_o is Kähler if and only if $d\omega_o = 0$.

Definition 1 (Gill) *The* ***Chern-Ricci flow*** *starting at* ω_o *is* $\frac{\partial}{\partial t}\omega(t) = Ric(\omega(t))$ *with initial condition* $\omega(0) = \omega_o$ *where*

$$Ric(\omega) = -\sqrt{-1}\partial\overline{\partial} \log \det g$$

is called the ***Chern-Ricci form****.*

A few remarks, if ω_o is Kähler then the Chern-Ricci flow is the same as the Kähler-Ricci flow. Our focus then is going to be on non-Kähler flows. Other flows of Hermitian metrics can be found in work by Streets-Tian. Let's parse out the differences between Kähler and non-Kähler manifolds:

1. Note that if M is a projective algebraic variety then it admits a Kähler metric.
2. There are certainly lots of non-Kähler manifolds (admit no Kähler metric). An example of this is a Hopf surface

$$M = (\mathbb{C}^2 \setminus \{0\})/[(z^1, z^2) \sim (2z^1, 2z^2)]$$

   where here M is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$ by the mapping $z \to \left(\frac{z}{|z|}, |z|\right) \in \mathbb{S}^3 \times (\mathbb{R}^+/[x \sim 2x])$. It's immediate that $H^2(M) = 0$ and so must be non-Kähler.

We would like to talk about the long-time existence of non-Kähler-Ricci flows. To recap, if $\omega(t)$ solves the Kähler-Ricci flow $\frac{\partial}{\partial t}\omega = -Ric(\omega)$, then we can consider the cohomology classes

$$[\omega(t)] \in H^{(1,1)}(M, \mathbb{R}) = \{\text{closed real (1,1)-forms}\}/\mathrm{Im}\,\overline{\partial}.$$

We then see that $\frac{d}{dt}[\omega(t)] = -[Ric(\omega)] = -c_1(M)$ hence the chomology classes determine the flow $[\omega(t)] = [\omega_o] - tc_1(M) > 0$ as long as the flow exists. Results from Cao, Tsuji and Tian-Zhang shows that there exists a unique maximal solution to the Kähler-Ricci flow on $[0, T)$ where

$$T = \sup\{t > 0 \mid [\omega_o] - tc_1(M) > 0\}.$$

It's natural to ask whether or not there is an analagous result for the Chern-Ricci flow. The answer is yes.

Theorem 1 (Tosatti-Weinkove) *There exists a unique maximal solution to the Chern-Ricci flow on* $[0,T)$ *where*

$$T = \sup\left\{t > 0 \mid \exists\, \psi \in C^\infty(M)\ s.t.\ \omega_o - tRic(\omega_o) + \sqrt{-1}\partial\overline{\partial}\psi > 0\right\}$$

Let us discuss complex surfaces. It is convenient to work with certain choices of ω_o. Let M be a compact complex surface (i.e. $\dim_{\mathbb{C}} M = 2$).

Theorem 2 (Gauduchon) *Given a Hermitian metric* ω*, there exists* $\sigma \in C^\infty(M)$ *unique up to an additional constant such that*

$$\partial\overline{\partial}(e^\sigma \omega) = 0$$

where we identiy $\omega_o = e^\sigma\omega$*. We call* ω_o *of this form a* ***Gauduchon metric****.*

It is important to note that this $\partial\bar{\partial}$-closed property is preserved by the flow. First assume that M^2 is **minimal**, meaning that there are no (-1) curves. We would like to know what happens to the Chern-Ricci flow for (M, ω_o). We define the Kodaira dimension $Kod(M) = 2, 1, 0$, or $-\infty$ of M by the following: we consider the dimension of the space of holomorphic sections which scales for $l \gg 0$:

$$\dim H^0(M, K_M^l) \sim l^{Kod(M)}$$

Roughly speaking $Kod(M) = 2$ is analagous to $Ric < 0$, $Kod(M) = 0$ is analagous to $Ric = 0$, $Kod(M) = -\infty$ corresponds to some positive curvature, and $Kod(M) = 1$ means you have directions of negative and possibly zero curvature.

For $Kod(M) = 2$ it is known that M is algebraic and hence admits a Kähler metric. Nevertheless it is interesting to consider the case where we start with ω_o a non-Kähler metric. A result by Tosatti-Weinkove, Gill shows that Chern-Ricci flow exists for all t and $\frac{\omega(t)}{t} \to \omega_{KE}$ where ω_{KE} is a (singular) Kähler-Einstein metric where convergence is C^∞_{loc} over $M \setminus \bigcup_j C_j$ where C_j are (-2) curves, along with $Ric(\omega_{KE}) = -\omega_{KE}$ on $M \setminus \bigcup C_j$. The Kähler case of this was due to Cao, Tsuji, and Tian-Zhang. This completes the case of Kodaira dimension 2.

For Kodaira dimension 1, an example is a curve of high genus (the base) and take a product with a torus (the fiber). We expect the flow, after renormalizing, to shrink the fibers and converge to a metric on the base. This turns out to be more or less the only thing we get for non-Kähler case. Indeed if M is non-Kähler then M is an elliptic bundle or has a finite cover which is an elliptic bundle. A result by Tosatti-Weinkove-Yang shows that the Chern-Ricci flow exists for all t and

$$\frac{\omega(t)}{t} \to \omega_s$$

where ω_s is the pullback of the Kähler-Einstein metric on the base where convergence is C^0 and happens exponentially. This is essentially collapsing the fibres onto the base. The Kähler case was proven by Song-Tian.

For Kodaira dimension 0, we either obtain Calabi-Yau manifolds or Kodaira surfaces. The Calabi-Yau manifolds are Kähler while the Kodaira surfaces are non-Kähler. Gill proved that the Chern-Ricci flow exists for time with $\omega(t) \to \omega_\infty$ where $Ric(\omega_\infty) = 0$. The Kähler case was proven by Cao (using estimates of Yau).

The classification of surfaces with $Kod(M) = -\infty$ is still unknown and is an area of active research. The non-Kähler surfaces with $Kod(M) = -\infty$ are called **class VII surfaces**. Typically we look at the second Betti number. If $b_2 = 0$ we can get either Inoue surfaces or Hopf surfaces:

1. For Inoue surfaces, these tend to be quotients of $H \times \mathbb{C}$ fiber over $\mathbb{S}^1$, where H is the upper half plane. A result by Fang-Tosatti-Weinkove-Zheng shows that after an initial conformal change $\frac{\omega(t)}{t} \to \omega_\infty$ where ω_∞ is a multiple of the Poincare metric. The corresponding manifolds converge in the Gromov-Hausdorff sense $\left(M, \frac{\omega(t)}{t}\right) \to \mathbb{S}^1$.

2. For Hopf surfaces, Tosatti-Weinkove proved that there is blow-up in finite time where $Vol_M\omega(t) \to 0$ as $t \to T$. For an explicit example, we can start with

$$M = (\mathbb{C}^2 \setminus \{0\})/[z \sim 2z]$$

and take the metric

$$\omega_o = \frac{\sqrt{-1}}{|z^1|^2 + |z^2|^2}\delta_{ij} dz^i \wedge d\bar{z}^j.$$

In this specific example, the metrics converge to a degenerate but smooth nonnegative $(1,1)$ form. The kernel defines a non-integrable distribution which generates the tangent space of $\mathbb{S}^3$. In the Gromov-Hausdorff sense, $(M, \omega(t)) \to \mathbb{S}^1$.

When the second Betti number $b_2 > 0$ we have that the flow blows up in finite time, but we don't yet know anything more.

So far we have studied minimal surfaces, what about the non-minimal case?

Theorem 3 (Tosatti-Weinkove) *Suppose that M admits a (-1) curve and $Kod(M) \neq -\infty$. Then there exists a solution to the Chern-Ricci flow on $[0,T)$ for $T < \infty$, there exists a map $\pi : M \to N$ blowing down finitely many disjoint (-1) curves E_j, and $\omega(t) \to \omega_T$ where convergence is $C^\infty_{loc}(M \setminus \bigcup_j E_j)$, where ω_T is a smooth metric on $M \setminus \bigcup_j E_j$.*

We conjecture that one can continue the flow on N. In the Kähler case, this is due to Song-Weinkove and is the two-dimensional case of the analytic minimal model program of Song-Tian.