Global Solutions of the Teichmüller Harmonic Map Flow

Notes on talk of Peter Topping

May 6, 2016

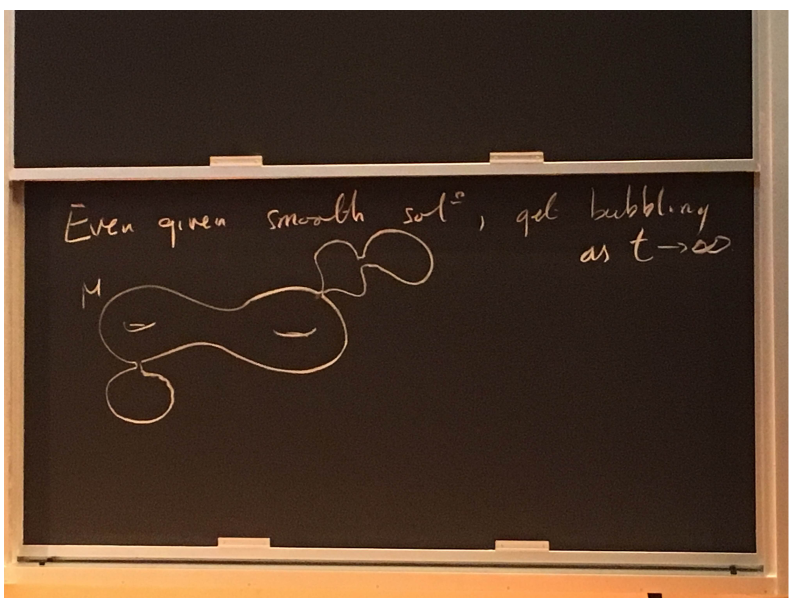
The work presented today is joint work with M. Rupflin. The discussion will be about nonlinear flow that may develop singularities. Consider the case where we have a map $u : (M, g) \to (N, G)$ between Riemannian manifolds. Let's discuss the energy of the map given by

$$E(u) = \frac{1}{2}\int_M |du|^2 d\mu_g$$

Throughout our discussion we will assume M to be a closed manifold, as well as N. The **harmonic map flow** is an L^2-gradient flow, and measures essentially how stretched the map u is. The flow for u in this case is given by

$$\frac{\partial u}{\partial t} = \tau_g(u) = \text{“}\Delta u\text{”}$$

where the quotations indicate that this operator is essentially a Laplacian. The work by Struwe in '85, he was able to construct global weak solutions to this flow. The solutions are smooth except for finitely many singularities in spacetime (where bubbling can occur). Even given a smooth solution, we in general get bubbling as $t \to \infty$. We get a limiting harmonic map and get what is called a *bubble tree.*



As we will see soon the bubbles turn out to be branched minimal immersions. Coming back to the energy functional we can view this as both a function of the map but also the metric of the domain. In other words we can write $E(u, g)$. The critical points of this new functional are going to be weakly conformal harmonic maps which are called **branched minimal immersions** or BMIs for short. The **Teichmüller map flow** is taken as the gradient flow on $E(u,g)$. We will restrict g to lie in the space M_c of metrics with constant curvature of either $-1,0,1$, and constant area one in the middle case.

Recall that

$$T_g M_c = \{\mathcal{L}_X g\} \oplus \{Re(\Psi) : \Psi \text{ is holomorphic quadratic differential}\}.$$

where X is any smooth vector field and $\mathcal{L}_X$ is the Lie derivative with respect to X. The latter tangent vectors are the so-called horizontal tangent vectors. Let

$$\mathcal{A} = (\{maps\} \times M_c)/\{\text{diffeomorphisms isotopic to } Id\}$$

A tangent vector in $\mathcal{A}$ can be represented by a variation of u and g with the variation of g horizontal. At this point we can take the L^2 inner product to define our gradient flow. The flow is given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= \tau_g(u) \\ \frac{\partial g}{\partial t} &= Re(\Psi(t)). \end{aligned}$$

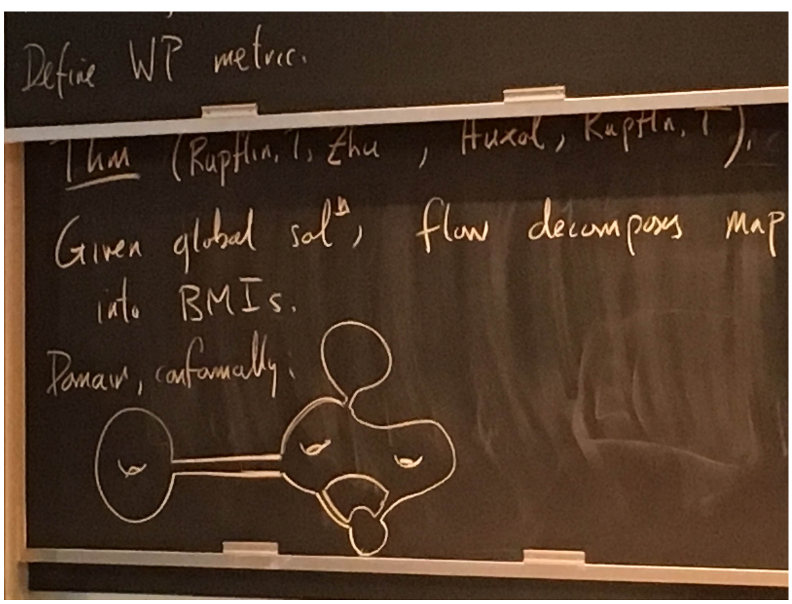
Let's compare this to the mean curvature flow, which would yield the same equation for the map, but the difference is with the metric, given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= \tau_g(u) \\ g &= u^* G. \end{aligned}$$

It's reasonable to ask at this point if the flow finds branched minimal immersions.

Theorem 1 (Rupflin-Topping-Zhu, Huxol-Rupflin-Topping) *Given a global solution, the flow decomposes the map u into branched minimal immersions.*

We would like to view the domain conformally, say as a genus 4 surface like the following:



At large times, the flow converges on each portion to a BMI. When we do this we have no loss of energy. The theorem above is guaranteed when we have a global solution, but when are we guaranteed a global solution?

Theorem 2 (Rupflin-Topping) *If (N, G) has non-positive curvature, then for every map and every metric (u_0, g_0) there exists a smooth global solution.*

What happens in the case of a general target space? In other words, what happens when the target manifold does not have non-positive curvature. A result from Rupflin shows the existence of solutions with bubbling until such time when the domain degenerates. This is behavior that could never happen for surfaces of genus 0 or 1. When this degeneration occurs the length of the shortest closed geodesic is going to zero. More formally if $l(t) \to 0$ as $t \to T < \infty$ where $l(t)$ is the shortest length of all closed geodesics at time t. Today, we would like to continue the flow beyond time T, and we'd like to do this in a canonical way (independent of the manner in which we do this).

Theorem 3 (Rupflin-Topping) *Let M be a closed, oriented surface of genus ≥ 2. Let (u, g) be a solution to the Teichmüller flow for $t \in [0, T)$ with*

$$\liminf_{t \uparrow T} inj_{g(t)}(M) = 0.$$

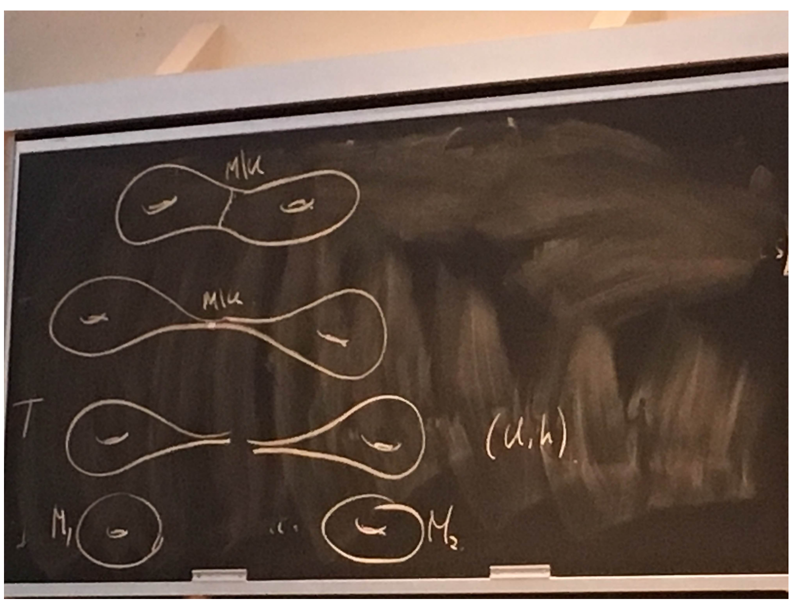
Then there exists an open set $U \subset M$ and hyperbolic metric h on U such that

1. $g(t) \to h$ *smoothly and locally on U as $t \uparrow T$,*
2. $\sup_{M \setminus U} inj_{g(t)}(\cdot) = 0$, *and*
3. $u(t) \to \overline{u}$ *smoothly and locally on $U \setminus \{bubble\ points\}$.*

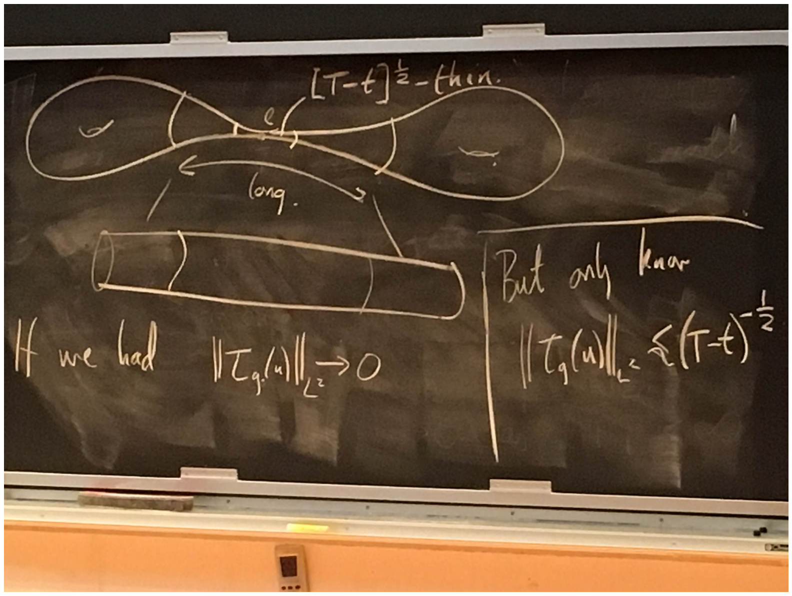
We have that (U, h) is conformally

$$(U, h) \simeq \coprod \{M_i\} / \{punctures\}$$

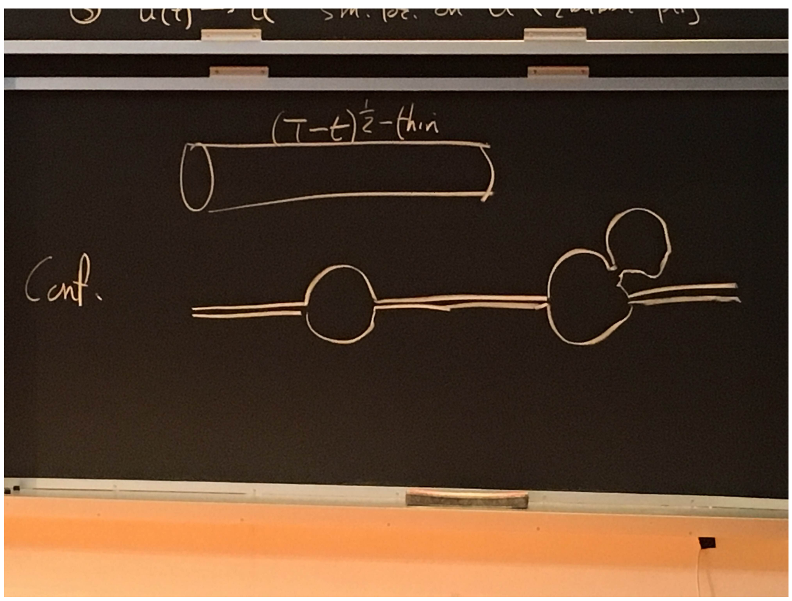
where the M_i are Riemann surfaces and the limit $\overline{u}$ extends to a collection of H^1-maps $\{u_i : M_i \to (N, G)\}$.



We would like to equip each M_i with a metric $g_i \in M_c$, then we would like to restart the flow on each M_i with initial data (u_i, g_i). The danger here is that we can lose nontrivial topology or part of the map down the collars. As the geodesic in a region gets shorter the region itself will get much longer and longer.



If we had that $\|\tau_{g_0}(u)\|_{L^2} \to 0$ we would be able to extract a bubble tree, but we only know (roughly) that $\|\tau_g(u)\|_{L^2} \leq (T-t)^{-1/2}$. However, when we scale the metric up from g to g_0, the tension scales down, and we can extract a bubble tree on the $[T-t]$-thin part. At this part we can desribe what is called a *bubble branch*.



What we want to avoid is losing energy down a collar just outside this region.

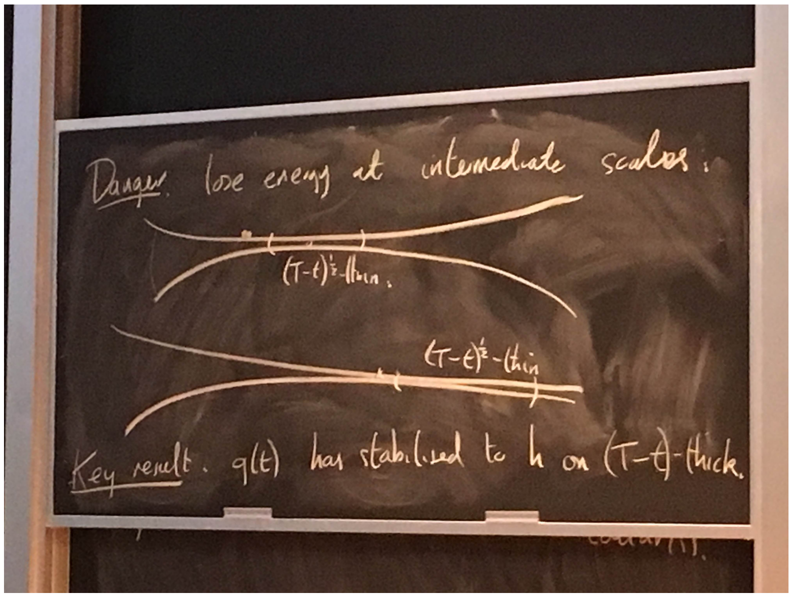
Theorem 4 (Rupflin-Topping) *No other energy is lost down a collar: Defining E_{thin} as the energy lost down a collar by*

$$E_{thin} \;:=\; \lim_{\delta\downarrow 0}\lim_{t\uparrow T} E(u(t), \delta - thin\,(M, g(t)))$$

we have in fact that

$$E_{thin} = \lim_{t \uparrow T} E(u(t), [T-t] - thin\,(M, g(t))).$$

In order to prove this, we need to acknowledge that the danger is that we could lose energy at intermediate scales. Dirty energy, i.e. parts of the map that never look harmonic and carry some nontrivial energy and topology might persist just outside the $(T-t)^{1/2}$-thin region. They would not be captured by the continued flow, and would be lost.



A key result to prove the theorem is that $g(t)$ has stabilized to h on the $[T-t]$-thick region.