A Gap Theorem and some Uniform Estimates for Ricci Flows on Homogeneous Spaces

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Everything discussed today is joint work with R. La Fuente and C. Böhm. We will be discussing homogeneous spaces (M^n, g) that have no boundary $\partial M^n = \varnothing$, they are complete and connected. For them to be homogeneous spaces we have that for all points $p, q \in M$ there is an isometry $f : M \to M$ where $f(p) = q$. We are also interested in studying local homogeneous spaces, which are related to homogeneous spaces. The latter differ from the former in the sense that they may be incomplete manifolds, and we now have local isometries. That is for every $p, q \in M$ there exists a map $f : B_\epsilon(p) \to B_\epsilon(q)$ where each ball is contained in M and f is an isometry with $f(p) = q$. We end up having quantities that are globally constant in this case. For instance the Riemannian curvature tensor satisfies $|Riem|_g : M \to \mathbb{R}$ is constant. We additionally have that $|\nabla Riem|_g : M \to \mathbb{R}$ is constant as well, but we have to be careful since there in general do not exist bounds on how large this quantity can be. For instance there exist examples where $|Riem|_g = 1$ but $|\nabla Riem|_g = N$ where we can take N as large as we like.

It's clear that homogeneous implies locally homogeneous, but the converse of this not necessarily true. It's natural to ask, if we have a locally homogeneous space M, does there exist a map $f : \Omega(\subseteq M) \to (N, g)$ where (N, g) is globally homogeneous and $f : \Omega \to f(\Omega)$ is an isometry? The answer is no actually. In work done by Kowalski and F. Lustaria and F. Tricerri they give examples of this fact. We do have however that if (M, g) is locally homogeneous, then it is also analytic. For instance if we have maps $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ are coordinates where $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ and $\varphi_\alpha \circ \varphi_\beta^{-1}$ is analytic, and the metric g with respect to this atlas is analytic.

Theorem 1 (Böhm, La Fuente, Simon) *If $(M^n, g(t))_{t\in[a,b]}$ is a homogeneous Ricci flow, then there exists a constant $0 < c(n) < \infty$ such that*

$$|Riem|_{g(b)} \leq c(n) \max\left\{\frac{1}{b-a}, R(g(b)) - R(g(a))\right\}.$$

Corollary 1 *Let $(M^n, g(t))$ be a maximal homogeneous solution on some interval I, then we have that*

1. *if $I = [0, T)$ with $T < \infty$ and $R(g(0)) = 1$, we have $|Riem|_{g(t)}(T - t) \in \left[\frac{1}{8}, c(n)\right]$ for all $t \in [\delta(n)T, T)$ where $0 < \delta(n) < 1$ and $c(n) < \infty$. Or,*
2. *if $I = [0, \infty)$ and $R(g(0)) = -1$ we obtain a Type III solution, where $|Riem|_{g(t)} t \in [0, c(n))$ for all $t \in I$. Or,*
3. *if $I = (-\infty, -1]$ and $R(g(-1)) = 1$ we have that $|Riem|_{g(t)}|t| \in [c(n), C(n)]$ where $c(n) > 0$.*

It is important to point out that C. Böhm that the first and second cases of the above corollary includes constants depending on the initial manifold (M, g_0). At the moment there are no statements on volume. We now proceed to the main ideas of the proof:

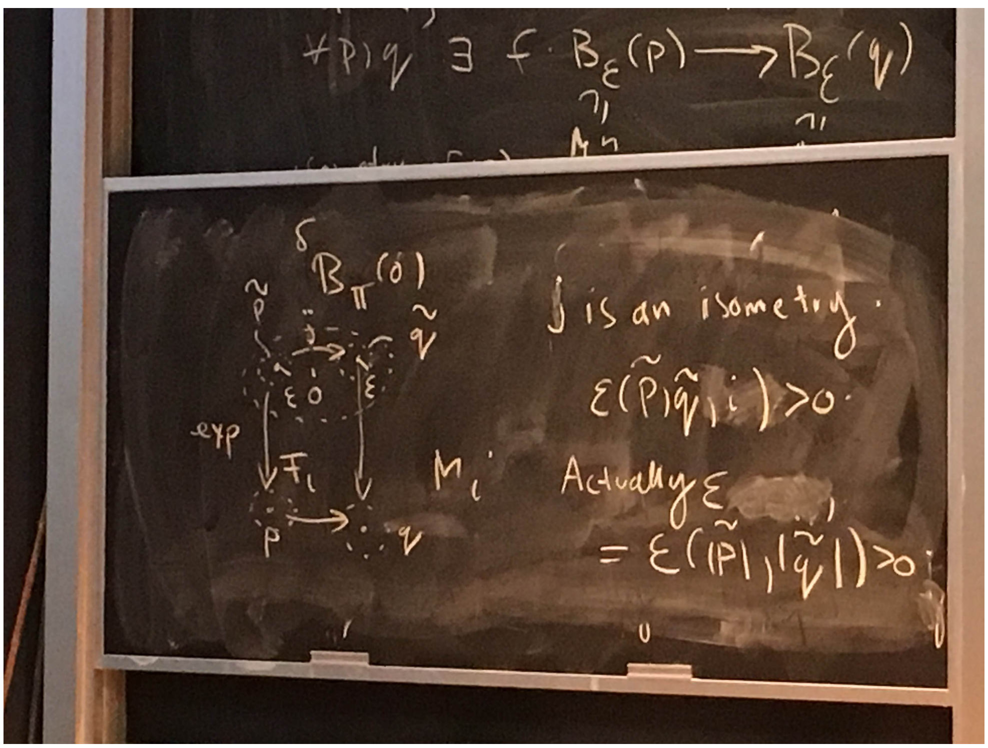
Proof: The main ingredients of the proof are the following:

1. The first ingredient comes form another theorem: For $n \in \mathbb{N}$ there exists $c(n) > 0$ such that if (M^n, g) is a homogeneous manifold then $|Riem|_g \leq c(n)|Ric|_g$. For example, there exists $\varepsilon(n) \in (0,1)$ such that the Weyl curvature $|Weyl|_g \leq (n - \varepsilon(n))|Riem|_g$.
2. This result comes from D. Alekseevski and N. Kimelfeld. In a homogeneous space we have that $|Ric|_g = 0$ implies that $|Riem|_g = 0$.
3. A result by A. Spiro made an improvement on the above statement, namely the same remains true in locally homogeneous spaces (i.e. $|Ric|_g = 0$ implies $|Riem|_g = 0$).

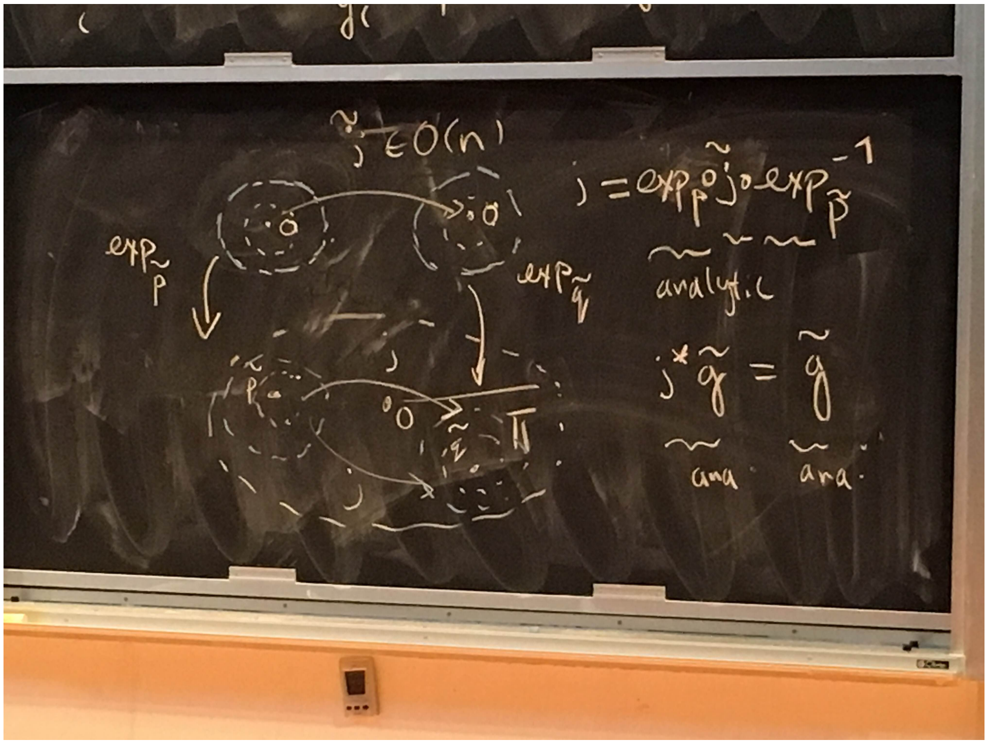
To prove the first item (the second theorem), assume that the result is false. Thus, suppose there exists a sequence of homogeneous spaces (M_i^n, g_i) such that

$$|Riem|_{g_i} \;\; \geq \;\; i|Ric|_{g_i}$$

for all $i \in \mathbb{N}$. Observe then that $|Ric|_{g_i} \leq \frac{1}{i}|Riem|_{g_i}$. Assume now that $|Riem|_{g_1} = 1$. Let $p_i \in M_i$ and use the exponential map $\mathrm{Exp}_{p_i} : T_{p_i}M_i \to M_i$ where we identify $T_{p_i}M_i \approx \mathbb{R}^n$ and $\mathrm{Exp}_{p_i}(0) = p_i$. Now we know this restricts to a diffeomorphism, and suppose we restrict to the ball of radius π, hence in other words $\mathrm{Exp}_{p_i}|_{{}^\delta B_\pi(0)}$, and this yields an immersion. We then see that we have a local cover $\left({}^\delta B_\pi(0), \widetilde{g}_i\right)$ where we have that $\widetilde{g}_i = f_i^* g_i$ and $1^{\delta}_{c(n)} \leq \widetilde{g}_i \leq c(n)\delta$ on ${}^\delta B_{\varepsilon(n)}(0)$.



At the moment we don't know if ε depends on $\tilde{p}, \tilde{q}$, or i. In fact we simply have that $\varepsilon = \varepsilon(\tilde{p}, \tilde{q}) > 0$, hence we can take a limit.



Essentially we have that if j is our local isometry in M, we can lift this to an isometry $\widetilde{j} \in O(n)$ such that $j = \text{Exp}_p \circ \widetilde{j} \circ \text{Exp}_{\widetilde{p}}^{-1}$ is analytic. We then obtain a Cheeger-Gromov limit for these balls $\left({}^{\widetilde{g}_i}B_\pi(0), \widetilde{g}\right) \to \left({}^{\widetilde{g}}B_\pi(0), \widetilde{g}\right)$ where $Ric(\widetilde{g}) = 0$ and $Riem \equiv 0$. Using the ideas of Cheeger-Anderson in $W^{1,p}$ harmonic coordinates we can evaluate the integral

$$0 \;<\; \delta(n) \;<\int_{{}^{\widetilde{g}_i}B_{\frac{\pi}{2}}(0)} |Riem|^{\frac{n}{2}} d\widetilde{g}_i \;\to\; \int_{{}^{\widetilde{g}}B_{\frac{\pi}{2}}(0)} |Riem|^{\frac{n}{2}} d\widetilde{g} \;=\; 0$$

which contradicts our assumption. □

For the proof of theorem 1 we let $K = |Riem|_{g(b)}$. If $\frac{1}{16K} \geq (b-a)$ we have that $K \leq \frac{1}{16(b-a)}$ hence

$$\int_a^b |Riem(t)|^2 dt \;=\; c(n)\int_a^b 2|Ric|^2 dt \;=\; c(n)\int \dot{R}(t)dt \;=\; c(n)(R(g(b)) - R(g(a))).$$

If $t \in \left[b - \frac{1}{16K}, b\right]$, then doubling our estimate we have that

$$|Riem(g(t))| \;\geq\; \frac{1}{2}|Riem(g(b))| \;=\; \frac{1}{2}K.$$

This implies that for some constant $\tilde{c}(n)$ we have that

$$\tilde{c}(n)(R(g(b)) - R(g(a))) \;\geq\; \int_a^b |Riem(g(t))|^2 \;\geq\; \int_{-\frac{1}{16K}}^b |Riem(g(t))|^2 \;=\; \frac{|Riem(g(b))|}{64}.$$

□