A Gap Theorem and some Uniform Estimates for Ricci Flows on Homogeneous Spaces

Miles Simon

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Everything discussed today is joint work with R. Lafuente and C. Böhm. We will be discussing homogeneous spaces (M^n, g) that have no boundary $\partial M^n = \emptyset$, they are complete and connected. For them to be homogeneous spaces we have that for all points $p, q \in M$ there is an isometry $f : M \to M$ where $f(p) = q$. We are also interested in studying local homogeneous spaces, which are related to homogeneous spaces. The latter differ from the former in the sense that they may be incomplete manifolds, and we now only have local isometries. That is for every $p, q \in M$ there exists a map $f : B_\epsilon(p) \to B_\epsilon(q)$ where each ball is compactly contained in M and f is an isometry with $f(p) = q$. We end up having quantities that are globally constant in both cases. For instance the Riemannian curvature tensor satisfies $|Riem|_g : M \to \mathbb{R}$ is constant. We additionally have that $|\nabla Riem|_g : M \to \mathbb{R}$ is constant as well, but we have to be careful since there in general do not exist bounds on how large this quantity can be. For instance there exist examples where $|Riem|_g = 1$ but $|\nabla Riem|_g = N$ where we can take N as large as we like.

It's clear that homogeneous implies locally homogeneous, but the converse of this not true : It's natural to ask, if we have a locally homogeneous space (M, g), does there exist a map $f : (\Omega(\subseteq M), g) \to (N, h)$ where (N, h) is globally homogeneous sucht that f is an isometry? The answer is no. In work done by O. Kowalski and F. Lastaria and F. Tricerri examples are constructed for which no such f exists. We do have however that if (M, g) is locally homogeneous, then it is also analytic. That is we have a covering of M by coordinate charts $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ where $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ and $\varphi_\alpha \circ \varphi_\beta^{-1}$ is analytic, and the metric g with respect to this atlas is analytic.

Theorem 1 (Böhm, Lafuente, Simon) *If $(M^n, g(t))_{t\in[a,b]}$ is a homogeneous Ricci flow, then there exists a constant $0 < c(n) < \infty$ such that*

$$|Riem|_{g(b)} \;\leq\; c(n) \max\left\{\frac{1}{b-a}, R(g(b)) - R(g(a))\right\}.$$

Corollary 1 *Let $(M^n, g(t))$ be a maximal homogeneous solution on some interval I, then we have that*

1. *If $I = [0, T)$ with $T < \infty$ and $R(g(0)) = 1$, we have $|Riem|_{g(t)}(T - t) \in \left[\frac{1}{8}, c(n)\right]$ for all $t \in [\delta(n)T, T)$ where $0 < \delta(n) < 1$.*
2. *If $I = [0, \infty)$ and $R(g(0)) = -1$ we obtain a Type III solution, where $|Riem|_{g(t)} t \in [0, c(n))$ for all $t \in I$.*
3. *If $I = (-\infty, -1]$ and $R(g(-1)) = 1$ we have that $|Riem|_{g(t)}|t| \in [c(n), C(n)]$ for some $0 < C(n) < \infty$.*

Remarks
C. Böhm proved results of the type given in the first and second case of the above corollary, but there the constants were dependent on the initial manifold (M, g_0) (not just n).
The volume does not appear in the statements above : for example a non-collapsing type assumption is not necessary.

We now proceed to the main ideas of the proof:

The main ingredient of the proof of Theorem 1 is the following Theorem, which has nothing to do with Ricci flow :

Theorem 2 (Böhm, Lafuente, Simon) *For $n \in \mathbb{N}$ there exists $c(n) > 0$ such that if (M^n, g) is a homogeneous manifold then $|Riem|_g \le c(n)|Ric|_g$. This is equivalent to : there exists $\varepsilon(n) \in (0,1)$ such that the Weyl curvature satisfies $|Weyl|_g \le (1 - \varepsilon(n))|Riem|_g$.*

The main ingredients of the proof of this Theorem are :

(a) A result of A. Spiro : In a locally homogeneous space we have that $|Ric|_g = 0$ implies that $|Riem|_g = 0$. In the homogeneous case this was proved by D. Alekseevski and N. Kimelfeld.

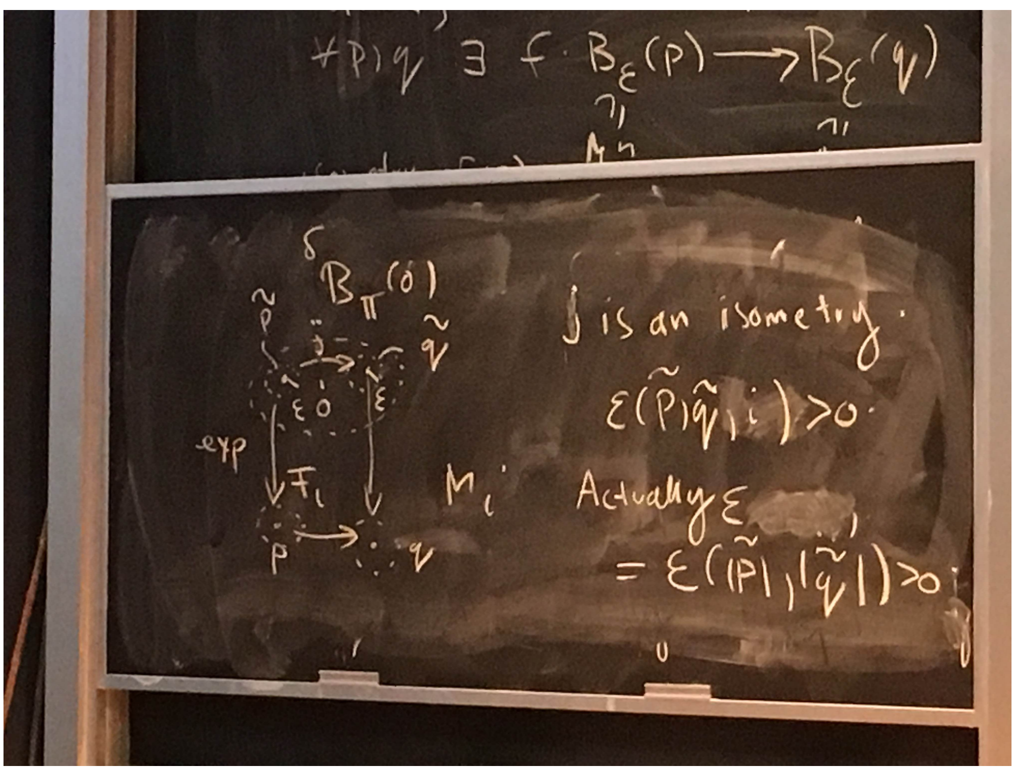
(b) A convergence result for Riemannian spaces similar to one proved by J. Cheeger and M. Anderson.

To prove Theorem 2, assume that the result is false. That is, suppose there exists a sequence of homogeneous spaces (M_i^n, g_i) such that

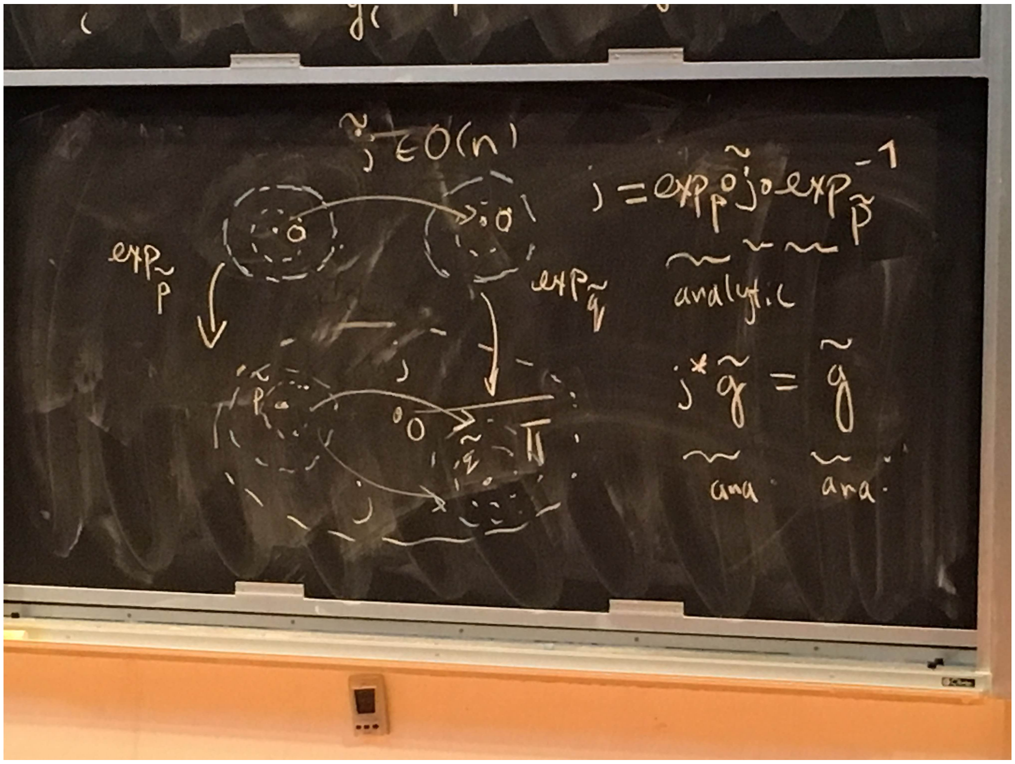
$$|Riem|_{g_i} \;\; \ge \;\; i|Ric|_{g_i}$$

for all $i \in \mathbb{N}$. Observe then that $|Ric|_{g_i} \le \frac{1}{i}|Riem|_{g_i}$. Scale so that $|Riem|_{g_i} = 1$ and $|Ric|_{g_i} \le \frac{1}{i}$. Let $p_i \in M_i$ and consider the exponential map $\mathrm{Exp}_{p_i} : T_{p_i}M_i \to M_i$ where we identify $T_{p_i}M_i \approx \mathbb{R}^n$ and $\mathrm{Exp}_{p_i}(0) = p_i$. If we restrict to the ball of radius π, that is we consider $\mathrm{Exp}_{p_i}|_{{}^\delta B_\pi(0)} :\to M$, this map is then (in view of the Rauch comparison theorems) an immersion. We then see that we have a local cover $\left({}^\delta B_\pi(0), \widetilde{g}_i\right)$ where $\widetilde{g}_i = f_i^* g_i$ satisfies $\frac{1}{c(n)}\delta \le \widetilde{g}_i \le c(n)\delta$ on ${}^\delta B_{\alpha(n)}(0)$ and $\mathrm{Inj}(\tilde{g}_i)(0) = \pi$, where the second last fact follows from $|\tilde{Riem}| = 1$ and the Jacobi equations. In particular, $Vol({}^{\tilde{g}_i}B_{\alpha(n)}(0)) \ge v_0(n) > 0$ and hence the sequence $({}^{\tilde{g}_i}B_\pi(0) = {}^\delta B_\pi(0), \tilde{g}_i)$ is non-collapsing.

Using the fact that $\mathrm{Exp}_{p_i}|_{{}^\delta B_\pi(0)} : {}^\delta B_\pi(0) \to M$ is an immersion we see the following : For all $\tilde{p}, \tilde{q} \in {}^{\tilde{g}_i}B_\pi(0)$ there exists an $\varepsilon > 0$ such that $\mathrm{Exp}|_{B_\varepsilon(\tilde{p})}$ and $\mathrm{Exp}|_{B_\varepsilon(\tilde{q})}$ are diffeomorphisms onto their images. Using the fact that there is an isometry in M which takes the point $p = \mathrm{Exp}(\tilde{p})$ to $q = \mathrm{Exp}(\tilde{q})$, we see by composing this map with $\mathrm{Exp}|_{B_\varepsilon(\tilde{p})}$ and $(\mathrm{Exp}|_{B_\varepsilon(\tilde{q})})^{-1}$ that there is an isometry from $B_\varepsilon(\tilde{p}) \subseteq {}^{\tilde{g}_i}B_\pi(0)$ to $B_\varepsilon(\tilde{q})$ taking $\tilde{p}$ to $\tilde{q}$. We call this isometry $j : B_\varepsilon(\tilde{p}) \to B_\varepsilon(\tilde{q})$.



At the moment ε depends on $\tilde{p}, \tilde{q}$ and i. In fact we simply have that $\varepsilon = \varepsilon(|\tilde{p}|, |\tilde{q}|) > 0$, as we now explain.



We consider the exponential map at $\tilde{p}$ and the exponential map at $\tilde{q}$. Lifting j, using these maps, we obtain a map, $\tilde{j}$, which is an element in $O(n)$. That is $j = \mathrm{Exp}_{\tilde{q}} \circ \tilde{j} \circ \mathrm{Exp}_{\tilde{p}}^{-1}$ and $j^*\tilde{g} = \tilde{g}$ (we have dropped the index i) on $B_\varepsilon(\tilde{p})$ (there is a typo in the picture above: it should be $j = \mathrm{Exp}_{\tilde{q}} \circ \tilde{j} \circ \mathrm{Exp}_{\tilde{p}}^{-1}$ and **not** $j = \mathrm{Exp}_p \circ \tilde{j} \circ \mathrm{Exp}_{\tilde{p}}^{-1}$). Using the fact that all maps and metrics are analytic, we can extend this map j to an isometry on a ball of radius $\delta = \delta(\max(\pi - |\tilde{q}|, \pi - |\tilde{p}|))$. That is $\varepsilon = \varepsilon(|\tilde{p}|, |\tilde{q}|) > 0$, as claimed. That is : the size of the balls where we have local isometries is not degenerating with i. We then obtain a smooth Cheeger-Gromov limit $\left({}^{\widetilde{g}_i}B_\pi(0), \widetilde{g}\right) \to \left({}^{\widetilde{g}}B_\pi(0), \widetilde{g}\right)$ where $Ric(\widetilde{g}) = 0$ and $|Riem| \leq 1$ such that the limiting space is locally homogeneous. Using the ideas of Cheeger-Anderson on $W^{1,p}$ harmonic coordinates we show

$$0 \;<\; \delta(n) \;<\int_{{}^{\widetilde{g}_i}B_{\frac{\pi}{2}}(0)} |Riem|^{\frac{n}{2}} d\widetilde{g}_i \;\to\; \int_{{}^{\widetilde{g}}B_{\frac{\pi}{2}}(0)} |Riem|^{\frac{n}{2}} d\widetilde{g} \;=\; 0$$

□

For the proof of theorem 1 we let $K = |Riem|_{g(b)}$. If $\frac{1}{16K} \geq (b-a)$ we have that $K \leq \frac{1}{16(b-a)}$ as claimed. So assume $\frac{1}{16K} < (b-a)$. Then

$$\int_a^b |Riem(t)|^2 dt \;\leq\; c(n)\int_a^b 2|Ric|^2 dt \;=\; c(n)\int \dot{R}(t)dt \;=\; c(n)(R(g(b)) - R(g(a))).$$

If $t \in \left[b - \frac{1}{16K}, b\right]$, then from the so called *doubling estimate* we have that

$$|Riem(g(t))| \;\geq\; \frac{1}{2}|Riem(g(b))| \;=\; \frac{1}{2}K.$$

This implies

$$c(n)(R(g(b)) - R(g(a))) \;\geq\; \int_a^b |Riem(g(t))|^2 \;\geq\; \int_{b-\frac{1}{16K}}^b |Riem(g(t))|^2 \;\geq\; \frac{|Riem(g(b))|}{64}.$$

□