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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Emily Norton **Email/Phone:** nebenem@gmail.com

Speaker's Name: Vera Serganova

Talk Title: Combinatorics, representations and geometry of algebraic supergroups

Date: 8 / 28 / 14 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Lie superalgebras, Kac modules, BGG reciprocity, blocks, weight diagrams, cap diagrams, multiplicity 1 flag

Please summarize the lecture in 5 or fewer sentences: Intro to rep theory of Lie superalgebras, focusing on $\mathfrak{gl}(m|n)$ and a version of category O for it. Using combinatorial diagrams, Serganova gives a criterion for ~~two~~ when two blocks are equivalent, as well as a theorem that decomposition numbers for simples in Kac modules (=Vermas) are 0 or 1, and a combinatorial rule for when the multiplicity is 1.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>**Computer Presentations**</u>: Obtain a copy of their presentation
  - <u>**Overhead**</u>: Obtain a copy or use the originals and scan them
  - <u>**Blackboard**</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - <u>**Handouts**</u>: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to <u>notes@msri.org</u> with the workshop name and your name in the subject line.

① Combinatorics, representations and geometry of algebraic supergroups, I

Vera Serganova, 8/28/14

Lie superalgebras $\mathbb{Z}_2$ graded, sign rule $\mathfrak{g} = \underset{\text{even}}{\mathfrak{g}_0} \oplus \underset{\text{odd}}{\mathfrak{g}_1}$, $[\,,\,]: \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$

$[x,y] = (-1)^{\bar{x}\bar{y}}[y,x]$ $\quad \bar{x}, \bar{y}$ = parity

Jacobi identity: $[x,[y,z]] = [[x,y],z] + (-1)^{\bar{x}\bar{y}}[y,[x,z]]$

1977: V. Kac studied simple Lie s.alg (=superalg) / $\mathbb{C}$ rep theory, observed reps aren't completely reducible

$$\mathfrak{gl}(m|n) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; \begin{matrix} A \; m\times m & C \; n\times m \\ B \; m\times n & D \; n\times n \end{matrix} \right\}$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\} \quad [x,y] = xy - (-1)^{\bar{x}\bar{y}} yx$$

supertrace: $\operatorname{str} X = \operatorname{tr} A - \operatorname{tr} D$

$\mathfrak{sl}(m|n) := \{ X \in \mathfrak{gl}(m|n) \mid \operatorname{str} X = 0 \}$

$m \neq n \Rightarrow \mathfrak{sl}(m|n)$ is simple

$m = n \Rightarrow \operatorname{str} \mathbb{1} = 0$, $\mathfrak{sl}(m|n)$ has nontrivial center $\mathbb{C}\cdot\mathbb{1}$, doesn't split

Algebraic supergroups G alg gp, then $\mathbb{C}[G]$ is a Hopf algebra.

Super-version: G alg s.gp, $A = \mathbb{C}[G]$ is a Hopf superalgebra,

$A = A_0 \oplus A_1$, $\Delta: A \to A\otimes A$ the comultiplication, and have sign rule:

$$(a\otimes b)(c\otimes d) = (-1)^{\bar{b}\bar{c}}\, ac\otimes bd$$

Harish-Chandra pairs $\mathfrak{g}$ Lie s.alg, G_0 alg gp with Lie $G_0 = \mathfrak{g}_0$

A rep of G_0 on $\mathfrak{g}_1$, inf version is given by $[\mathfrak{g}_0, \mathfrak{g}_1]$

Thm (Masuoka) $(\mathfrak{g}, G_0)$ Category of HC pairs $\simeq$ Cat. of alg s.gp's

From now on, we're interested in:

Rep G = category of finite-dim reps of an alg s.gp G.

= $\mathfrak{g}$-modules integrable over G_0

Let G_0 be a reductive algebraic group.

$\operatorname{Rep} G_0 \longrightarrow \operatorname{Rep} G$ induction functor

$M \longmapsto U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M$

note: PBW $\Rightarrow \operatorname{Gr} U(\mathfrak{g}) = S(\mathfrak{g}) = S(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)$

Facts
- Rep G has enough projectives
- Rep G is a Frobenius category, i.e. every projective module is injective $\Rightarrow$ can do coinduction, and:

$$\operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} M = \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} (M \otimes \Lambda^{top} \mathfrak{g}_1)$$

$G = GL(m|n)$ $\quad \mathfrak{g} = \mathfrak{g}^- \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}^+}_{\text{parabolic } \mathfrak{p}}$ $\quad \mathfrak{g}^+ = \left\{\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\right\}$, $\mathfrak{g}^- = \left\{\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\right\}$

G_0 reps $\longleftrightarrow \Lambda^+ = \left\{ (a_1, \dots, a_m \mid b_1, \dots, b_n) \;\middle|\; a_i, b_j \in \mathbb{Z},\ a_1 \geq a_2 \geq \dots \geq a_m,\ b_1 \geq b_2 \geq \dots \geq b_n \right\}$

$\Lambda^+ \ni \lambda$

$L^0(\lambda) :=$ simple G_0-rep w/ highest weight λ

$K(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L^0(\lambda)$ "Kac module" (like Verma)

$K(\lambda)$ has a unique simple quotient $L(\lambda)$ and all simple modules arise in this way. When λ is in "generic position" (λ lies outside the union of certain hyperplanes) then $K(\lambda)$ is simple.

$P(\lambda) :=$ projective cover of $L(\lambda)$

We get a highest weight category $\Rightarrow$ $P(\lambda)$ has a filtration whose quotients $K(\mu)$ satisfy BGG reciprocity:

$$[P(\lambda) : K(\mu)] = [K(\mu) : L(\lambda)]$$

(which says: the multiplicity of $K(\mu)$ in $P(\lambda)$ equals the multiplicity of $L(\lambda)$ in $K(\mu)$)

③ **Rmk** The point of doing all this – we want to write the character of $L(\lambda)$. This was done by Serganova, and later by Brundan in a different way.

Rmk This highest weight category w/ Kac modules etc has infinite cohomological dimension.

Technique: weight diagrams (Brundan, Stroppel)

$$\rho = (m, m-1, \dots, 1 \mid -1, -2, \dots, -n)$$

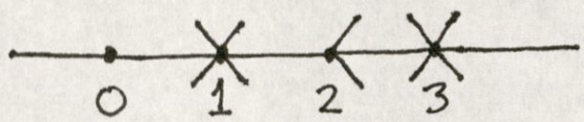
$$\lambda + \rho = (c_1 > \dots > c_m \mid d_1 > \dots > d_n)$$

The weight diagram is integer # line where in position c_i you put $>$ and in position $-d_i$ you put $<$:

for example, take $GL(2|3)$, V its natural rep, $\lambda = (1\,0 \mid 0\,0\,0)$

$\lambda + \rho = (3\ 1 \mid -1\ -2\ -3)$

then the weight diagram f_λ is



If the weight diagram f_λ has no crosses ($= \times$) then $K(\lambda) = L(\lambda)$.
The more crosses in f_λ, the more trouble!

Blocks $U(\mathfrak{g}) \supset Z(\mathfrak{g})$ center. In the super case, $Z(\mathfrak{g})$ isn't Noetherian.

Sergeev: $$Z(\mathfrak{g}) \simeq \frac{\mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]}{\left(f(x_1, \dots, x_i + t, \dots, x_m, y_1, \dots, y_i + t, \dots, y_n) - f(x_1, \dots, x_m, y_1, \dots, y_n) \right)}$$

Central characters $\chi : Z(\mathfrak{g}) \to \mathbb{C}$

$\mathrm{Rep}\, G = \bigoplus \mathrm{Rep}_\chi G$, where $\mathrm{Rep}_\chi G$ = all modules admitting central character χ

Question When do two modules lie in the same block?

$$\underset{\text{weight}}{\lambda} \rightsquigarrow \underset{\text{weight diagram}}{f_\lambda} \rightsquigarrow \underset{\text{core}}{c_\lambda} = f_\lambda \text{ with all } \times \text{ removed}$$

Prop $L(\lambda)$ and $L(\mu)$ lie in the same block if they have the same core: $c_\lambda = c_\mu$.

$\#X$ in f_λ = "degree of atypicality" of the block containing $L(\lambda)$

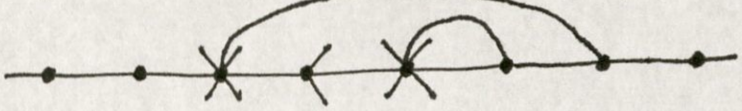
Every λ in the block has the same number of crosses in its f_λ.

Thm (Serganova) Two blocks of Rep $GL(m|n)$ and Rep $GL(m'|n')$ are equivalent iff those blocks have the same degree of atypicality (= same # crosses).

Study of blocks can be reduced to the $GL(p|p)$ case.

Now: would like to study $[K(\lambda):L(\mu)] = [P(\mu):K(\lambda)]$.

To do this, use cap diagrams. Equip weight diagram f_λ w/ caps:



There's a unique way to put caps on f_λ following the rules:

- every cap has X at left end
- there's no empty position under a cap
- all X's are involved
- caps do not cross

Thm (Brundan, independently, Serganova) $[P(\lambda):K(\mu)] = 0$ or 1, and $[P(\lambda):K(\mu)] = 1$ iff μ is obtained from λ by moving some X's of λ along ~~the~~ their caps in f_λ.

$\mathrm{Ext}^1(L(\lambda), L(\mu)) = 0$ or $\mathbb{C}$, and it's $\mathbb{C}$ iff λ is obtained from μ (or vice versa) by moving a single X along its cap.

$= \mathrm{Ext}^1(L(\mu), L(\lambda))$

Rmk When the block has just one X then the block is tame, i.e. we can describe its indecomposables. If the degree of atypicality of the block is > 1, then the block is wild.