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# NOTETAKER CHECKLIST FORM

**(Complete one for each talk.)**

**Name:** Emily Norton **Email/Phone:** nebenem@gmail.com

**Speaker's Name:** Julianna Tymoczko

**Talk Title:** Springer representations and other geometric representations

**Date:** 8 / 28 / 14 **Time:** 11 : 00 am / pm (circle one)

**List 6-12 key words for the talk:** Flag, Springer fiber, nilpotent cone, Springer resolution, Springer representations on $H^{top}$(Springer fibers)

**Please summarize the lecture in 5 or fewer sentences:** Geometric representation theory aims to show a geometric representation exists, while combinatorial representation theory works to characterize such a representation. Springer's construction of irreducible representations of the symmetric group on the top cohomology of Springer fibers illustrates geometric representation theory, whereas combinatorics comes into play in character formulas and connections to Catalan numbers.

## CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>**Computer Presentations**</u>: Obtain a copy of their presentation
  - <u>**Overhead**</u>: Obtain a copy or use the originals and scan them
  - <u>**Blackboard**</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - <u>**Handouts**</u>: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

# Springer representations and other geometric representations, I

Julianna Tymoczko, 8/28/14

**Springer rep's** Today: geometry + combinatorics
tomorrow: future directions

**Geom rep theory** goal: establish that some geometric rep exists
**Combinatorial rep theory** characterize the rep

**Springer variety**

**Def** A flag is a nested collection of complex vector spaces

$$V_1 \subseteq V_2 \subset \dots \subset V_n = \mathbb{C}^n \quad , \quad \text{each } V_i \text{ is } i\text{-dim'l}$$

Given an $n \times n$ matrix $X$, consider all flags fixed by $X$:

$$S_X = \{\text{Flags } V_\bullet : XV_i \subset V_i \ \forall i\} \qquad (\text{"Springer fiber"})$$

**Examples** 1) $X = (0)$ $\quad S_X = \{\text{all flags } V_\bullet\}$

2) $X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $\quad XV_1 \subseteq V_1$ so $V_1$ spanned by e-vector for $X$
$\Rightarrow$ by one of $e_1, e_2, e_3$
$XV_2 \subseteq V_2$ so $V_2$ spanned by 2 e-vectors

$$S_X = \{\text{permutation flags}\}$$

3) $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad XV_1 \subseteq V_1 \Rightarrow V_1 \subseteq \ker X = \langle e_1 \rangle$
$XV_2 \subseteq V_2 \Rightarrow V_2 \subseteq \ker X^2 = \langle e_1, e_2 \rangle$

regular nilpotent

$S_X$ consists of a single flag

4) $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad XV_1 \subseteq V_1$ so $V_1 \subseteq \ker X = \langle e_1, e_3 \rangle$
choice $V_1 = \langle e_1 + ae_3 \rangle$ or $V_1 = \langle e_3 \rangle$

subregular nilpotent

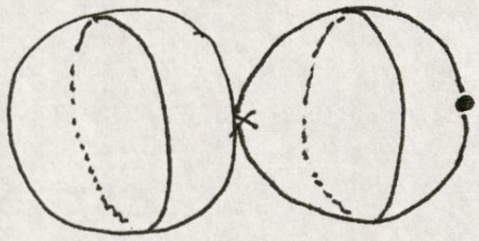
$XV_2 \subseteq V_2$ For this $X$ have $\operatorname{Im} X = \langle e_1 \rangle$
choice $V_2 = \ker X$ or $V_1 = \langle e_1 \rangle$
$V_2 = \langle e_1, e_2 + be_3 \rangle$

② Picture:

$V_1 = \langle e_1 \rangle$ $\qquad$ $V_1 = \langle e_1 + a e_3 \rangle$ $\qquad$ $V_1 = \langle e_3 \rangle$

$V_2 = \langle e_1, e_2 + b e_3 \rangle$ $\qquad$ $V_2 = \langle e_1, e_3 \rangle$ $\qquad$ $V_2 = \langle e_1, e_3 \rangle$

all components have the same dimension



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<u>Redescribe Springer varieties</u>: describe $V_\bullet$ as a matrix: $g$ a matrix whose first $i$ columns span $V_i$ ($\forall i$). $B$ = upper $\Delta$ invertible matrices. All matrices in $gB$ represent the same flag.

$$S_X = \{ \text{flags } gB \mid Xg \in g\mathfrak{b} \} \qquad \mathfrak{b} = \text{upper } \Delta \text{ matrices}$$
$$= \{ \text{flags } gB \mid g^{-1} X g \in \mathfrak{b} \}$$

Consider $(X, gB) \in \mathfrak{g} \times G/B$ and project to 1st, 2nd factors.

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[d, "S"'] & \subseteq & \mathfrak{g} \times G/B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{N} & \subseteq \mathfrak{g} & & G/B \end{tikzcd}$$

$\tilde{\mathcal{N}} = S^{-1}(\mathcal{N})$
$= \{ (X, gB) \mid X \text{ nilpotent}, g^{-1} X g \in \mathfrak{b} \}$

$\mathcal{N} = \{\text{nilpotent matrices}\}$

$S = \pi_1(\pi_1^{-1}(\mathcal{N}) \cap \pi_2^{-1}(G/B))$

- preimage of $X$ under Springer map $S$ is $S_X$ (Springer fiber)
- preimage of $gB$ under $\pi_2$ is:
$$\pi_2^{-1}(gB) = \{ (X, gB) \mid g^{-1} X g \in \mathfrak{b} \} = \{ X \text{ in } g\mathfrak{b}g^{-1} \} =: \tilde{\mathfrak{g}}$$
- fiber of $\pi_2^{-1}$ smooth, base $G/B$ smooth, so $\tilde{\mathcal{N}}$ smooth
- $$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[d, "\pi_1"] & \subset & \tilde{\mathfrak{g}} \arrow[d, "\pi_1"] \\ \mathcal{N} & \subset & \mathfrak{g} \end{tikzcd}$$

<u>Facts</u>
- <u>generic</u> elt of $\mathfrak{g}$ is regular semisimple (distinct e-values)
- fiber is $n!$ permutation flags
- fiber carries an action of $S_n$

③ <u>Point of Springer theory</u>: can move rep of $S_n$ on generic fiber over to special nilpotent fibers.

The Springer rep has been constructed in many ways by many ppl: Springer, Kazhdan-Lusztig, Borho-MacPherson, Kashiwara, De Concini-Procesi, Lusztig, . . .

Properties of Springer's rep on $H^*(S_x)$:

1) $H^{top}(S_x)$ is an irrep of $S_n$
2) all irreps of $S_n$ arise this way
3) these irreps are unique up to conj class of $X$

( <u>Rmks</u> • certain constructions / ideas recur, e.g. Springer fibers are cotangent bdl's..
• can generalize outside type $A_{n-1}$ )

<u>In type $A_{n-1}$</u>:

4) the ungraded rep on $H^*(S_\mu)$ is the $S_n$-rep on the Young subgroup $S_{\mu_1} \times S_{\mu_2} \times \dots \subseteq S_n$.
Here $\mu$ = Jordan type of $X$ = partition of $n = (\mu_1, \mu_2, \dots)$
<u>not'n</u> $S_\mu$ = Springer fiber but $S_{\mu_i}$ = symmetric group on $\mu_i$ letters.

We have a character formula:

$$\text{Poincaré poly}(S_\mu) = \sum_\lambda \chi^\lambda K_{\lambda\mu}(q)$$

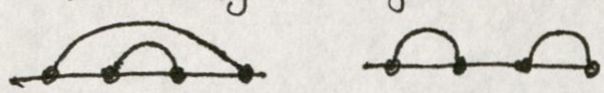
$\chi^\lambda$ character of $S_n$
$K_{\lambda\mu}$ $q$-Kostka poly

<u>Moral of next ex'l</u> the universe is all interconnected, dude. (or: Catalan #'s r promiscuous!)

<u>Ex'l</u> Catalan numbers. 1) count #(2-row standard tableaux on $(n,n)$)

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ rows, col's increase l. to r., top to bottom

2) non-crossing matchings on $1, 2, \dots, 2n$



<u>Obs.</u> (Fung, Khovanov) Each component in Springer fiber $S_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline\end{array}}$ is an iterated fiber bundle of $\mathbb{P}^1$'s

Tableaux and n.c.m.'s descriptions of Cat.#'s describe different aspects of this bundle:

<u>tableaux</u> tell what order to throw in basis vectors
<u>n.c.m.'s</u> encodes $\mathbb{P}^1$'s in bundle