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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Emily Norton **Email/Phone:** nebenem@gmail.com

Speaker's Name: Vera Serganova

Talk Title: Combinatorics, representations and geometry of algebraic supergroups

Date: 8 / 28 / 14 **Time:** 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: translation functors, categorification, super flag manifold, BGG reciprocity, support varieties

Please summarize the lecture in 5 or fewer sentences: First, looks at translation functors and categorification in type A. Second, ~~intro~~ introduces other Lie superalgebras, for which Kac modules don't exist. A virtual module, called Euler characteristic, replaces Kac module and satisfies BGG reciprocity. Third, the lecture concludes with support varieties which give rise to a nice tensor functor in type A. It is conjectured that this functor takes semisimples to semisimples.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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Combinatorics, representations and geomerry of algebraic supergroups, II

Vera Serganova, 8/28/14

Methods 1. Translation functors and categorification

2. BWB theory

3. Support variety theory (method from modular rep theory)

1. $G = GL(m|n)$. $V = \mathbb{C}^{m|n}$, V^* natural and conatural reps

$M \mapsto M \otimes V$
$M \otimes V^*$ translation goes from one block to another:

translation functor $\rightarrow$ $T_{\chi,\eta} : Rep_\chi G \rightarrow Rep_\eta G$ (endofunctor if $\eta = \chi$)

$$M \longmapsto M \otimes V \xmapsto[pr_\eta]{} (M \otimes V)_\eta$$

$K(\lambda) \otimes V$ has filtration by $K(\mu)$

$\mathcal{K}^+ \subset \mathcal{K}(Rep\ G)$ generated by $[K(\lambda)]$

$\otimes V$ $>$ one position to right $\quad$ $\otimes V^*$ $>$ one position to left

$<$ one position to left $\quad$ $<$ one position to right

Ex'l $K(\times < \times) \otimes V = K(< \times \times) + K(\times < < >)$
$+ K(< > < \times)$

Note can't move a symbol to a spot that's already occupied by that same symbol.

If you move between blocks w/same # X's, or move to a block w/more X's, PIM $\longmapsto$ PIM (PIM = proj indec module)

Categorification of $\mathfrak{gl}(\infty)$ (Brundan) $\quad \mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(n)$

E, E^* natural + conatural reps. Dynkin diagram:

$$\cdots \cdots \!-\!\!\!\underset{\alpha_0}{\bullet}\!\!\!-\!\!\!\underset{\alpha_1}{\bullet}\!\!\!-\!\!\!\underset{\alpha_2}{\bullet}\!\!\!-\; \cdots\cdots$$

$$\mathcal{K}^+_{\mathbb{C}} = \mathcal{K}^+ \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow[\varphi]{} \Lambda^m E \otimes \Lambda^n E^*$$

$$[K(\lambda)] \longmapsto e_\lambda$$

②

$E : \{e_i\}_{i\in\mathbb{Z}}$ $\Big\rangle$ bases

$E^* : \{f_j\}_{j\in\mathbb{Z}}$

$\langle e_i, f_j\rangle = -\delta_{ij}$

$e_{c_1}\wedge\ldots\wedge e_{c_m}\otimes f_{d_1}\wedge\ldots\wedge f_{d_n}$ where $\lambda+\rho = (c_1,\ldots,c_m \mid d_1,\ldots,d_n)$

Under φ,

block $(\mathcal{K}^+_\chi)_{\mathbb{C}} \longrightarrow$ Weight space in $\Lambda^m E \otimes \Lambda^n E^*$

Serre-Chevalley generators of $\mathfrak{gl}_\infty$: $E_{i,i+1}$ $E_{i,i-1}$

$$\mathcal{E}_{i,i+1} = \bigoplus T_{\chi,\chi+d_i} \qquad \mathcal{E}_{i+1,i} = \bigoplus T^*_{\chi+d_i,\chi}$$

$$\varphi\circ\mathcal{E}_{i,i+1} = E_{i,i+1}\circ\varphi, \quad \varphi\circ\mathcal{E}_{i+1,i} = E_{i+1,i}\circ\varphi$$

Canonical basis $\{l_\lambda\}$ (Lusztig, Kashiwara) satisfies triangularity condition:

$$l_\lambda = e_\lambda + \sum_{\mu<\lambda} c_{\mu\lambda} e_\mu$$

φ takes projectives to canonical basis: $\varphi[P(\lambda)] = l(\lambda)$

To get irred modules on RHS, must complete then take dual basis. Then get decomp #'s from coeff's in canonical basis expressions:

$$[P(\lambda) : K(\mu)] = c_{\mu\lambda}$$

And we know how to calculate the $c_{\mu\lambda}$.

In other classical types: orthogonal + symplectic are glued together. $V = V_0 \oplus V_1$ fix b, non-degen. symm. form (even)

$$b(v,w) = (-1)^{\bar{v}\bar{w}} b(w,v)$$

Then $b|_{V_1}$ is a symplectic form

$\mathfrak{g} = \mathfrak{osp}(m|2n)$ $\mathfrak{g}_0 = \mathfrak{o}(m)\oplus\mathfrak{sp}(2n)$

③ another one: $p(n)$. $\dim V_0 = \dim V_1 = n$. Consider β, odd symm form.

$\mathfrak{g} = p(n)$ is the Lie alg. preserving β:

$$p(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \;\middle|\; B^t = B,\ C^t = -C \right\}$$

Rep G is <u>not</u> a highest-weight category! And Kac module doesn't exist...

Substitute for Kac module: constructed using super-geometry:

<u>Supergeometry</u> flag supermanifold G/B

Ex'l for $GL(1|2)$: $\mathbb{C}^{1|0} \subset \mathbb{C}^{1|1} \subset \mathbb{C}^{1|2}$

or $\mathbb{C}^{0|1} \subset \mathbb{C}^{0|2} \subset \mathbb{C}^{1|2}$

Supermanifold = usual flag mfld G_0/B_0 + a sheaf on it.

$$G/B \;\rightleftarrows\; G_0/B_0$$

<u>splitting</u> <u>exists</u> for $GL(m|n)$ ~~when~~ for specific choice of B only.

↳ then $K(\lambda) = H^0(G/B, \mathcal{O}(-\lambda))^*$

<u>Open question</u> calculate $H^i(G/B, \mathcal{O}(-\lambda))$ for B Borel supergroup in G.

<u>Do know</u> Euler characteristic:

$$\mathcal{E}(\lambda) = \sum (-1)^i [H^i(G/B, \mathcal{O}(-\lambda))]$$

$\mathcal{E}(\lambda)$ is an analog of $K(\lambda)$.

⇝ <u>Weak BGG reciprocity</u>

$$[P(\lambda) : \mathcal{E}(\mu)] = [\mathcal{E}(\mu) : L(\lambda)]$$

multiplicity in K^+

Using translation functors, can calculate decomp #'s for $osp(m|2n)$

The decomp #'s can be $-1, 0$, or 1.

Can have -1 bec there is odd cohomology.

interesting problem: understand this using derived categories.

Also: consider all osp together. Categorifies Fock space? "free bosons" categorify parabolic induction. $osp(m|2n) \supset p \supset osp(m-2|2n)$. G/P, irr P sheaf $L_P(\lambda)$, $H^i(G/P, L_\lambda(P))$ are known.

④ 2005, Serganova + Duflo

$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ Lie superalg. "Cell commuting cone" in odd part:

$$X = \{ x \in \mathfrak{g}_1 \mid [x,x] = 0 \} \text{ cone in } \mathfrak{g}_1$$

Note! $[x,x] = 2x^2$, so this is a nontrivial condition.

M – $\mathfrak{g}$-module s.t. $x^2 M = 0$.

Define $M_x = \mathrm{Ker}_* \, x / \mathrm{Im}\, x$.

If $M = \mathfrak{g}$, adjoint rep, then $\mathfrak{g}_x = \mathrm{Ker}\,\mathrm{ad}\,x / \mathrm{Im}\,\mathrm{ad}\,x$

$\rightsquigarrow$ functor $\mathfrak{g}\text{-mod} \xrightarrow[F_x]{} \mathfrak{g}_x\text{-mod}$

Nice functor, e.g. preserves superdimension: $\mathrm{sdim}\, F_x M = \mathrm{sdim}\, M$

additive: $F_x(M \oplus N) = F_x M \oplus F_x N$

preserves duality: $F_x(M^*) = (F_x M)^*$

tensor functor: !! preserves $\otimes$: $F_x(M \otimes N) = F_x M \otimes F_x N$

Define "associated variety":

$$X_M = \{ x \in X \mid F_x M \neq 0 \}$$

Properties of X_M:
1) Zariski closed cone
2) $X_{M \oplus N} = X_M \cup X_N$
3) $X_{M \otimes N} = X_M \cap X_N$
4) X_M is G_0-invariant
5) if M is projective in Rep G (assume G_0 is reductive) then $X_M = \{0\}$. In many cases the reverse implication is true.

Ex'l $G = GL(m|n)$

$$X = \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid ab = ba = 0 \right\} / G_0 \longleftrightarrow (p,q) \quad \begin{array}{l} p = \mathrm{rk}\, a \\ q = \mathrm{rk}\, b \end{array}$$

finitely many G_0-orbits

⑤ $X = \coprod X_k$ where $X_k = \{x \in X \mid \mathrm{rk}\, x = k\}$

$x, y \in X_k \Rightarrow \mathfrak{g}_x = \mathfrak{g}_y \simeq \mathfrak{gl}(m-k \mid n-k)$

$\rightsquigarrow$ tensor functor $\mathrm{Rep}\, GL(m|n) \longrightarrow \mathrm{Rep}\, GL(m-k|n-k)$

<u>Prop.</u> If $M \in \mathrm{Rep}_\chi\, GL(m|n)$, $x \in X_k$, $r =$ degree of atypicality $= \#\,\times$'s in f_λ

$F_x M \in \mathrm{Rep}_{\chi'} GL(m-k, n-k)$, $\#\chi' = r-k$

If degree of atypicality of M is k then $X_M \subset \overline{X_k}$

If $M = L(\lambda)$ irred then $X_M = \overline{X_k}$

Generic $L(\lambda)$ is in a block by itself. Then $\#\lambda = k$, $\mathrm{rk}\, x = k$,

$F_x L(\lambda) = L(\bar{\lambda}) \otimes \mathbb{C}^{a|b}$

multiplicity

<u>Conjecture</u> either a or b is 0

<u>Main conjecture</u> $F_x(\text{semisimple}) = \text{semisimple}$

If true, calculations of $F_x M$ for all M follow.

$\#\chi \leq 2$