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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Emily Norton **Email/Phone:** nebenem@gmail.com

Speaker's Name: Marie-France Vignéras

Talk Title: Pro-p-Iwahori-Hecke algebras of p-adic groups

Date: 8 / 29 / 14 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: pro-p-Iwahori-Hecke algebra, alcove walk basis, admissible representations, parabolic induction, supercuspidal modules

Please summarize the lecture in 5 or fewer sentences: This talk introduces the objects in the title, gives generators and relations for them, ~~and~~ and describes a nice basis, then goes on to relate their representation theory to that of p-adic groups. Study of the pro-p-Iwahori-Hecke algebra sheds light on questions about admissible and supercuspidal representations of the underlying p-adic group G. Recent results in this direction are surveyed, + open questions posed.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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Pro-p-Iwahori-Hecke algebras of p-adic groups

Marie-France Vignéras 8/29/14

1. Structure of $\mathcal{H}(G, I(1)) = \mathcal{H}(W(1), q_s, c_s)$
2. Supercuspidal representations

G-red p-adic gp $\quad G = G(F)$ red conn alg gp

$F = \mathbb{Q}_p$ or $\mathbb{F}_p((x))$

Modulo p reps of G

$G \longrightarrow \mathrm{Aut}\, V$ $\quad$ v.s./$\overline{\mathbb{F}}_p$ $\quad$ every $v \in V$ fixed by some open compact sb-gp.

$I(1)$ pro-p Iwahori sb-gp of G

$V \neq 0 \Rightarrow V^{I(1)} \neq 0$

$\mathcal{H}(G, I(1))$ = double cosets of G mod $I(1)$, convolution ring

$\mathrm{Mod}_{\overline{\mathbb{F}}_p}(G) \xrightarrow[\text{invariant functor}]{\text{pro-p Iwahori}} \mathrm{Mod}\, \mathcal{H}_{\overline{\mathbb{F}}_p}(G, I(1))$

This functor important bec G is hard, can't deform, but RHS easier, can deform it

T = max split F-torus in G

Z = centralizer of T

N = normalizer of T

Note All such tori are conjugate. likewise, w/ Iwahori's, so Hecke alg doesn't depend on choice of $I(1)$

Bruhat - Tits $\quad G = I(1) N I(1)$ $\quad N/Z_0(1) = W(1) \simeq I(1)\backslash G / I(1)$

$N \cap I(1) = Z_0(1)$ $\quad I(1) n_w I(1) \longleftrightarrow T_w \in \mathcal{H}(G, I(1))$

R - commutative ring

Thm $\mathcal{H}_R(G, I(1))$ = R-module basis $(T_w)_{w \in W(1)}$ with product satisfying relations:

Braid relations $\quad T_w T_{w'} = T_{ww'}$ $\quad w, w' \in W(1)$ and $\ell(ww') = \ell(w) + \ell(w')$

- to make sense of this, need to explain what length $\ell(w)$ means for $w \in W(1)$ $\rightsquigarrow$

(2) Length for $W(1)$:

There exists an affine Coxeter system $(W^{aff}, S^{aff}) \circlearrowleft \Omega$, f.g. comm gp

Z_k k-basis, k = residue field of $F = \mathbb{F}_q$

have s.e.s.: $1 \to Z_k \to W(1) \to W^{aff} \rtimes \Omega \to 1$

set $\ell(\Omega) = 0 = \ell(Z_k) \rightsquigarrow$ get length on $W(1)$.

<u>Quadratic relations</u> write $q_s = [I(1) n_s I(1) : I(1)]$ power of q

$T_s^2 = q_s s^2 + c_s T_s$ $\quad s \in S^{aff}(1)$

$\downarrow$

Z_k $\quad Z_{k,s} \subset Z_k$ $\quad c_s = \dfrac{q_s - 1}{|Z_{k,s}|} \sum_{t \in Z_{k,s}} t$

$c_{st} = c_s t$ $\quad t \in Z_k$, $s \in S^{aff}(1)$

<u>Generalization</u> $H_R(W(1), q_s, c_s)$ $\quad q_s \in R$, $c_s \in R[Z_k]$

$q_{st} = q_s$

$c_{st} = c_s$ $\quad t \in Z_k$

$s' = w s w^{-1}$ $\quad w \in W(1)$, $s' \in S^{aff}(1)$

$q_{s'} = q_s$

$c_{s'} = w c_s w^{-1}$

<u>Ex'l</u> $GL(2, D)$, D a division alg

$T = \begin{pmatrix} F^\times & 0 \\ 0 & F^\times \end{pmatrix}$ $\quad Z = \begin{pmatrix} D^\times & 0 \\ 0 & D^\times \end{pmatrix}$ $\quad N = ZU\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$I(1) = \{ g \in GL(2, \mathcal{O}_D)$ $\quad g = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod P_D$

$Z_0(1) = \begin{pmatrix} 1 + P_D & 0 \\ 0 & 1 + P_D \end{pmatrix}$ $\quad Z_k = \begin{pmatrix} k_D^\times & 0 \\ 0 & k_D^\times \end{pmatrix}$

<u>Prop</u> $\mathcal{H}_{R[\underline{q}_s^{\pm 1/2}]}(W(1), \underline{q}_s, c_s) \simeq \mathcal{H}_{R[\underline{q}_s^{\pm 1/2}]}(1, \underline{q}_s^{\pm 1/2}, c_s)$

$q_w = s_1 \cdots s_n$ $\quad$ if $w \in W(1)$ $\quad w = s_1 \cdots s_n u$ $\quad s_i \in S^{aff}(1)$, $u \in \Omega(1)$, n as small as possible

$c_s = w c_s w^{-1}$ $\quad s \in S^{aff}(1)$ $\quad w \in W(1)$

③ Alcove Walk Basis

Thm For any Weyl chamber $\mathcal{D}$ of $\Phi(G,T)$ there exists a basis $(E_{\mathcal{D}}(w))_{w\in W(1)}$ of $\mathcal{H}_R(W(1), q_s, c_s)$ s.t.

root system

- $E_{\mathcal{D}}(w)\, E_{w^{-1}(\mathcal{D})}(w') = (q_w q_{w'} q_{ww'}^{-1})^{1/2} E_{\mathcal{D}}(ww')$ $\quad w, w' \in W(1)$
- $E_{\mathcal{D}}(w) = T_w \quad w \in \Omega(1)$
- $E_{\mathcal{D}}(s) \in \{T_s, T_s - c_s\} \quad s \in S^{aff}(1)$

Cor $A_{\mathcal{D}} :=$ R-module generated by $(E_{\mathcal{D}}(\lambda))_{\lambda\in\Lambda(1)}$ is a subalgebra of $\mathcal{H}_R(W(1), q_s, c_s)$, with Bernstein-Lusztig basis

$E_{\mathcal{D}}(s) = T_s \quad \forall s \in S(1), \quad S = W_0 \cap S^{aff}$ and Bernstein relations

$$E_{\mathcal{D}}(s(\lambda))E_{\mathcal{D}} - E_{\mathcal{D}}(s)E_{\mathcal{D}}(\lambda) = \sum_{k \text{ finite}} c_{T(k)} E_{\mathcal{D}}(\mu(k)\lambda) \in A_{\mathcal{D}}$$

(Here, $\Lambda(1)$ is defined by:

$$1 \longrightarrow \Lambda(1) = \frac{Z}{Z_0(1)} \longrightarrow W(1) = \frac{N}{Z_0(1)} \rightarrow \frac{N}{Z} = W_0 \rightarrow 1$$

and $T(k) \in \{W(1)\text{-conj of } S^{aff}(1)\}$ and $\mu(k) \in \Lambda(1)$).

Thm If R is Noetherian then $\mathcal{H} := \mathcal{H}_R(W(1), q_s, c_s)$ is a finitely generated module over its center $A_{\mathcal{D}}^{W_0(1)}$, and the center is a finitely generated R-algebra.

Cor If R is a field then any irreducible $\mathcal{H}$-module is finite-dimensional.

Cor Every irred. $\mathcal{H}_{\overline{\mathbb{F}}_p}(G, I(1))$-module is finite-dimensional.

(9) Now, suppose $R = \overline{\mathbb{F}}_p$.

<u>Def</u> V, rep of G, is called <u>admissible</u> if $V^{I(1)}$ is finite-dim.

<u>Q'n</u> Irreducible $\Rightarrow$ admissible?

- known only for $GL(2, \mathbb{Q}_p)$. But many things that work well for $GL(2)$ don't work well in general. For example, <u>in general</u>:

$$\mathrm{Mod}_{\overline{\mathbb{F}}_p} G \longrightarrow \mathrm{Mod}\, \mathcal{H}_{\overline{\mathbb{F}}_p}(G, I(1))$$

$$\text{irr} \longmapsto \text{not irr}$$

<u>but for $PGL(2, \mathbb{Q}_p)$</u>, $\left\{ \begin{array}{c} \text{reps gen'd} \\ \text{by } I(1)\text{-invariant} \\ \text{vectors} \end{array} \right\} \simeq \mathrm{Mod}\, \mathcal{H}_{\overline{\mathbb{F}}_p}(G, I(1))$

(results of Rachel Ollivier, R.O. + Peter Schneider)

Parabolic Induction

$P = MN$ $\quad \mathrm{Mod}_{\overline{\mathbb{F}}_p}(M) \xrightarrow{I_P^G} \mathrm{Mod}_{\overline{\mathbb{F}}_p}(G)$ induction functor

<u>Def</u> V, an admissible rep of G, is called <u>supercuspidal</u> if it's not a subgp of $I_P^G(V)$, V an irrep of M

induction functor for Hecke algebra:

$$\mathrm{Mod}\, \mathcal{H}_{\overline{\mathbb{F}}_p}(M, I_M) \xrightarrow[- \otimes_{\mathcal{H}_P(G)} \mathcal{H}(G)]{\mathbb{I}_P^G} \mathrm{Mod}\, \mathcal{H}_{\overline{\mathbb{F}}_p}(G, I(1))$$

(here, $\mathcal{H}(G) \supset \mathcal{H}_P(G) \simeq \mathcal{H}_P(M)[T_M^{-1}] = \mathcal{H}(M)$)

<u>Thm</u> (Ollivier) pro-p Iwahori invariant functor commutes with parabolic induction:

$$\left(I_P^G(V)\right)^{I(1)} = I_P^G\left(V^{I_M(1)}\right)$$

⑤ Q'n Supercuspidals of G?

Supercuspidals of $\mathcal{H}_{\overline{\mathbb{F}}_p}(G, I(1))$ were classified by Ollivier, Vignéras, Abe.

$$\mathcal{H}_{\overline{\mathbb{F}}_p}(W(1), q_s, c_s) = \underset{\substack{\cup \\ (T_s)_{s \in S^{aff}(1)}}}{\mathcal{H}_R^{aff}} \otimes_{R[Z_k]} R[\Omega(1)] \qquad T_s(T_s - c_s) = 0$$

<u>Thm</u> Irreps which contain a character of $\mathcal{H}^{aff}_{\overline{\mathbb{F}}_p}$,

$\chi \neq$ sign, trivial characters,

$\chi(T_s) \neq 0 \quad \forall s$

$\chi(q_s) = 0, \quad \chi(c_s) \neq 0 \quad \forall s$

are the supercuspidal modules.

<u>Ex'l</u> # supercuspidal irreps of $\mathcal{H}(GL(n,F))$ of dim'n n, w/action of $\mathrm{diag}(P_F)$

= # irr poly's, unitary, of deg n, in $\mathbb{F}_q[x]$ $\quad q = p^n$

= # irreps mod p of $\mathrm{Gal}(\overline{F}/F)$ of dim n, w/ det (Frobenius) fixed

functor: $\mathrm{Mod}_{\overline{\mathbb{F}}_p} \mathcal{H}(GL(n, \mathbb{Q}_p), I(1)) \longrightarrow \mathrm{Mod}^{f.d.} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

supercuspidal dim n $\quad \simeq \quad$ irr dim n