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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Emily Norton **Email/Phone:** nebenem@gmail.com

Speaker's Name: Julianna Tymoczko

Talk Title: Springer representations + other geometric representations, Part 2

Date: 8 / 29 / 14 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: cohomology of Springer varieties, Hessenberg varieties, monodromy representation, Poincaré polynomial, Kostka-Foulkes polynomial, monomial basis

Please summarize the lecture in 5 or fewer sentences: "Geometric representation theory is the best field ever." The talk attempts to argue this position by emphasizing how geometry + combinatorics benefit from each other in Springer theory.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

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Springer representations and other geometric representations Part 2

Julianna Tymoczko, 8/29/14

Last time (1) geom rep thy: establish geom reps (2) comb. rep thy: analyze reps

Springer reps illustrate interplay of (1) + (2)

This time directions

- some reps are built to solve a specific problem
- the fact of having a geom rep connected to many things often means there's more going on $\rightsquigarrow$ new questions

Questions (1A) establish combinatorially/computationally useful presentations of a geom rep. E.g. Springer rep is awesome but not good for computing a character.

Reminder Borel's presentation of coho of G/B:

$$H^*(G/B) \cong \mathbb{C}[x_1, \dots, x_n] / \mathbb{C}[x_1, \dots, x_n]^W_+ \quad \text{(W-invariant poly's w/no constant term)}$$

Have analogous presentation for Springer varieties in type A:

$$H^*(S_\mu) \cong \mathbb{C}[x_1, \dots, x_n] / I_\mu \quad \leftarrow \text{symmetric ideal}$$

History '81 Kraft: takes scheme-theoretic intersection

$\overline{C_\mu} \cap T$ — This is non-reduced; $\exists$ S_n-action on $A_\mu :=$ its coord ring. The action is the rep induced from the trivial rep on the Young subgroup of S_n associated to the partition $\mu = (\mu_1, \dots, \mu_r)$, $S_{\mu_1} \times \dots \times S_{\mu_r}$

closure of conj class of Jordan type μ

diag matrices

'81 de Concini + Procesi: $A_\mu \cong H^*(S_\mu)$

'82 Tanisaki: better generators for I_μ

'08 Biagoli-Faridi-Rosas: even better generators.

sketch Draw Young diagram for μ, then number up the columns (L to R) except for 1st row, and lastly number R to L on 1st row.

11	10	9	8
3	5	7	
2	4	6	
1			

Set $e_i(S) :=$ $\sum$ all monomials in S w/degree of each variable $\leq i$

variables

Then $I_\mu = \left\langle \left\{ e_i(S) \;\middle|\; \begin{array}{l} \text{every subset } |S| = j \text{ in } \{x_1, \dots, x_n\} \\ \text{where } j \text{ is top of column with } i\text{, for each } i \end{array} \right\} \right\rangle$

Example

6	5	4
2	3	
1		

$\{e_1(S) : \text{subsets } S \text{ of size } 6\} = \{e_1(x_1, \dots, x_6)\}$,

$\{e_2(S)\} = \{e_2(x_1, \dots, x_6)\}$, $\{e_3(S) : \forall \text{ subsets } |S| = 5\}$, etc

'94 Garcia-Procesi: monomial basis for $H^*(S_\mu)$. Contains info about cell structure of Springer fibers S_μ.

'11? Mbirika: relates to geometry of Springer fibers.

Q'n Other types?

(1B) When does a geom rep exist? • given rep, ∃ geom rep realizing it? Necessary / sufficient conditions? • given variety, does it admit a geom rep? Given family of varieties, if one of them admits rep, what about the others? e.g. Springer fibers belong to family of Hessenberg varieties which have an add'l parameter.

Hessenberg varieties: 2 formulations

① $h: \{1, 2, \ldots, n\} \circlearrowleft$ nondecreasing f'n, $h(i) \geq h(i-1)$ and $h(i) \geq i$

② a subspace $H \subseteq \mathfrak{g}$ s.t. $\mathfrak{b} \subseteq H$, $[H, \mathfrak{b}] \subseteq H$

Given such f'n h or space H, Hessenberg variety is:

$$\mathcal{H}(X, h) = \{\text{Flags } V_\bullet : XV_i \subseteq V_{h(i)}\ \forall i\} = \{\text{Flags } gB : g^{-1}Xg \in H\}$$

<u>Note</u> Springer case is $h(i) = i$, $H = \mathfrak{b}$

<u>Thm</u> (Macpherson-Tymoczko) There exists a Springer-like rep of S_n on $H^*(\text{Hess})$ satisfying the formula:

$$\text{Poincaré poly}(\mathcal{H}(X, h)) = \sum_{\lambda \geq \mu} R_H^\lambda \tilde{K}_{\lambda\mu}(q)$$

Here, X has Jordan type μ & is a nilpotent matrix, R_H^λ denotes λ^{th} isotypic component of S_n-rep on regular semisimple Hess var, and $\tilde{K}_{\lambda\mu}$ is normalized Kostka-Foulkes poly.

<u>Note</u> this rep isn't Springer rep, rather it's the monodromy rep

(2A) What explicitly are the characters of geom reps?

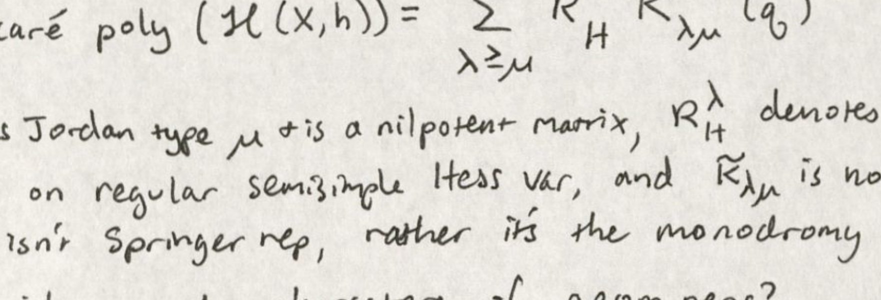
e.g. the S_n-rep on $\mathcal{H}(X, h)$ for reg ss X is mysterious

<u>Conjecture</u> The rep on $\mathcal{H}(X, H)$ for X reg ss is $\sum_{w \in H} p_{\mu(w)}$

where $\mu(w)$ = cycle type of w,

$p_k = x_1^k + \ldots + x_n^k$, $p_\mu = p_{\mu_1} \cdots p_{\mu_s}$ if $\mu = (\mu_1, \ldots, \mu_s)$

<u>Ex'ls</u> H:



$p_{(3)} = s_{(3)} + s_{(1,1,1)} + 2s_{(2,1)}$

triv, sign, Schur poly

! H need not be a parabolic

$p_{(3)} + 2p_{(1,1,1)} + 3p_{(2,1)} = 6s_{(3)}$

(2B) characterize geometry or representations via combinatorial model

ex'ls from yesterday: • in Serganova's talks, a rep of $\mathfrak{gl}(m|n)$ was simple, tame, or wild depending on # X's in weight diagram

• in T's talk, Springer fiber components had bundle structure described by NCM's/ YT's

<u>Qn/Ex'l</u> (n, n, n) Springer fibers: components are indexed by "webs"

Can this be extended to other types? to other rectangular shapes? to arbitrary μ?