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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Emily Norton **Email/Phone:** nebenem@gmail.com

Speaker's Name: Christine Huyghe

Talk Title: Survey over localization theorems for representation theory of Lie algebras

Date: 8 / 29 / 14 **Time:** 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: D-modules, Kazhdan-Lusztig conjecture, holonomic D-modules, localization theorems, characteristic variety

Please summarize the lecture in 5 or fewer sentences: Introduces D-modules, localization theorems and sketches the proof of Kazhdan-Lusztig conjectures.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Survey over localization theorems for representation theory of Lie algebras

Christine Huyghe, 8/29/14

G: alg ss gp$/\mathbb{C}$ $\quad \mathfrak{g} = \mathrm{Lie}\, G \quad B$: Borel $\quad \mathfrak{b} = \mathrm{Lie}\, B$

$U(\mathfrak{g})$: universal enveloping algebra $\quad W = N_G(T)/T \quad \rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$

1. Statement of Kazhdan-Lusztig conjectures

$w \in W$, M_w Verma module of highest weight $w(\rho) - \rho$

L_w unique simple quotient of M_w

K.L. defined poly's $\in \mathbb{Z}[q]$ for each pair of elt's $w_1, w_2 \in W$:
$P_{w_1, w_2}(q)$ $\quad w_0 :=$ longest elt of W

<u>Conj</u> 1) $$\mathrm{ch}(M_w) = \sum_{y \leq w} P_{w_0 w, w_0 y}(1)\, \mathrm{ch}(L_y)$$

2) $$\mathrm{ch}(L_w) = \sum_{y \leq w} (-1)^{\ell(w) - \ell(y)} P_{y,w}(1)\, \mathrm{ch}(M_y)$$

<u>Rmk</u> 1) $\Leftrightarrow$ 2)

2. $\mathcal{D}$-modules

$X/\mathbb{C}$ smooth alg. variety. $\mathcal{D}_X :=$ subring of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and the derivations of $\mathcal{O}_X$ $(=: \mathcal{T}_X)$.

Let $t_1, \ldots, t_N$ be local coords, then $\mathcal{D}_X = \bigoplus \mathcal{O}_X \partial_{t_1}^{k_1} \cdots \partial_{t_N}^{k_N}$.

$\mathcal{D}_X$ is filtered by order of differential operators:

$$\mathcal{D}_{X,n} = \bigoplus_{|k| \leq n} \mathcal{O}_X \partial_{\underline{t}}^{\underline{k}} \qquad \partial_{\underline{t}}^{\underline{k}} := \partial_{t_1}^{k_1} \cdots \partial_{t_N}^{k_N}$$

Associated graded: $\mathrm{gr}\, \mathcal{D}_X \simeq \mathcal{S}(\mathcal{T}_X)$ locally polynomial $(\simeq \mathcal{O}_X[\xi_1, \ldots, \xi_N])$

i.e. locally a poly ring in $2N$ variables.

$\rightarrow \mathcal{D}_X$ is coherent.

② If $X = \mathbb{A}^N_{\mathbb{C}}$, system of linear PDE $\sum_{j=1}^{q} \underset{\mathcal{D}_X}{\hat{P}_{ij}} u_j = 0$ (S)

$P_{ij} \in \mathcal{D}_X$, $1 \le i \le p$

Consider $\pi : \mathcal{D}_X^{\,p} \to \mathcal{D}_X^{\,q}$

$$(R_1, \dots, R_p) \longmapsto \left(\sum_{i=1}^{p} R_i P_{i1}, \dots, \sum_{i=1}^{p} R_i P_{iq}\right)$$

Then $\mathcal{M} \mathcal{D}_X^{\,q} / \mathrm{Ker}\, \pi$ is the $\mathcal{D}_X$-module attached to (S), and $\mathrm{Sol}(\mathcal{M}) = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$.

More generally, if ξ is a connection module over X (so, $\mathcal{O}_X$-module free of finite rank, $\nabla : \xi \to \xi \otimes \Omega^1$ satisfying Leibnitz rule) s.t. $\nabla \circ \nabla = 0$, then it defines a $\mathcal{D}_X$-module locally, $\partial_{t_i} \cdot e = \nabla(\partial_{t_i}) \cdot e$.

Now let X be a curve, $x \in X$, ξ a meromorphic connection module over $X \setminus \{x\}$, x defined locally by $t = 0$.

<u>Def</u> ξ has <u>regular singularities</u> ("<u>r. s.</u>") if locally ∇ can be written in some basis of ξ with a matrix with poles of order ≤ 1. "$du = \frac{1}{t} A(t) \cdot u(t) \otimes dt$" $A(t)$ w/coeffs in $\mathcal{O}_X$

<u>Holonomic Modules</u> Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module. Locally, there exists a filtration of $\mathcal{M}$ s.t. $\mathrm{gr}\, \mathcal{M}$ is a $\mathrm{gr}\, \mathcal{D}$-module of finite type. Since fin. type, can define:

$$J := \mathrm{Ann}_{\mathrm{gr}\, \mathcal{D}}\, \mathrm{gr}\, \mathcal{M}$$

Define $\sqrt{J}$, subvar. of $\mathrm{Spec}\, \mathrm{gr}\, \mathcal{D} = T^*X$

$$\mathrm{Spec}\, \mathrm{gr}\, \mathcal{D} \overset{\shortparallel}{=} \mathcal{S}(\mathcal{O}_X)$$

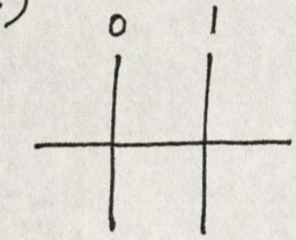
<u>Def</u> $\mathcal{M}$ is <u>holonomic</u> if $\dim \mathrm{char}\, \mathcal{M} = \dim X$.

characteristic variety

Note in general, $\dim X \le \dim \operatorname{char} \mathcal{M} \le 2 \dim X$.

Ex'l $X = \mathbb{A}^1$, $\mathcal{M} = \mathcal{D}_X / ((t-1)t^i \partial_t^a + \text{lower terms})$

Then $\operatorname{char} \mathcal{M}$ is defined by $(t-1)t\xi = 0$



Ex'l $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X t$

then $\operatorname{char} \mathcal{M}$ defined by $t = 0$

Holonomic

- $\mathcal{E}$ a connection module $\Rightarrow$ $\operatorname{char} \xi = 0$
- If $\mathcal{M}$ holonomic over X, smooth curve, then $\exists\, U \subset X$ open dense s.t. $\mathcal{M}|_U$ is a connection module.

We say $\mathcal{M}$ has r.s. if $\mathcal{M}|_U$ has r.s.

All this can be defined in higher dimensions $\to$ r.s. holonomic $\mathcal{D}$-modules. The categories of (r.s.) holonomic $\mathcal{D}$-modules are stable by usual cohomological operations.

- one defines, for $\mathcal{M}$ a $\mathcal{D}_X$-module,

$DR(\mathcal{M})$: $R\mathcal{H}om(\mathcal{O}_X, \mathcal{M}) = \text{coho}\,(0 \to \mathcal{M} \to \mathcal{M} \otimes \Omega^1_X \to \cdots \to \mathcal{M} \otimes \Omega^N_X \to 0)$

In particular $DR(X) = DR(\mathcal{O}_X)$.

Localization Theorems (BB, Br-Ka). $X = G/B$. $\mathcal{M}$: $\mathcal{D}_X$-module, M: $U(\mathfrak{g})$-module w/ trivial central character.

(i) The functors $\mathcal{M} \to \Gamma(X, \mathcal{M})$ and $M \to \mathcal{D}_X \otimes_{U\mathfrak{g}} M$ are inverse to each other.

(ii) $\forall i \ge 1$, $H^i(X, \mathcal{M}) = 0$

(4) Rmks (i) X G-homogeneous, $\exists\ \mathfrak{g} \to \mathcal{T}_X$ extending to $\mathcal{U}(\mathfrak{g}) \to \mathcal{D}_X$

$\Gamma(X, \mathcal{D}_X) \simeq \mathcal{U}\mathfrak{g} / \mathfrak{Z}_+$ $\quad \mathfrak{Z}_+ =$ elts of center of $\mathfrak{g}\,\mathcal{U}(\mathfrak{g})$

(ii) $\mathcal{O}(-2)$ over $\mathbb{P}^1$ can't be a $\mathcal{D}_{\mathbb{P}^1}$-module.

Idea of Br-Ka pf Let $\mathcal{H}$ = cat of s.r. hol. $\mathcal{D}$-mods whose char variety has support in the union of conormal bdls of Schubert varieties $X_w \subset X$. Set $\tilde{\mathcal{O}}_{triv} = \mathcal{U}\mathfrak{g}$-mod, $\mathfrak{n}$-locally finite, + trivial central character

(where $\mathfrak{n} \subset \mathfrak{b}$ is nilpotents)

Br-Ka prove: $\mathcal{H} \simeq \tilde{\mathcal{O}}_{triv}$ (equiv of categories)

Moreover (i) if $m \in \mathcal{H}$, $ch(\Gamma(X, m)) = \sum_{w \in W} (-1)^{\operatorname{codim} X_w} \chi_w(m)\, ch(M_w)$

where $\chi_w(m) := \sum (-1)^j \dim_{\mathbb{C}} \operatorname{Ext}^j_{\mathcal{D}}(\mathcal{O}_X, m)$

"de Rham characteristic"

(ii) M_w corresponds to the $\mathcal{D}_X$-dual of

$$\mathcal{H}^{\operatorname{codim} X_w}_{[X_w]}(\mathcal{O}_X)$$

(ii) is a difficult result of Kempf.

About the formula:

$$ch(L_w) = \sum_{w \in W} (-1)^{\operatorname{codim} X_w} \chi_w(\mathcal{D} \otimes L_w)\, ch(M_w)$$

$\Rightarrow$ one of the K.L. conj's

Key argument Let $m \in \mathcal{H}$, how to pass from results for $\mathcal{D} \otimes M_w$ to results for m? Proved by ind'n on $\#\{w \in W \mid X_w \subset \operatorname{Supp} m\}$

Pick w s.t. $w(\rho)$ is minimal w.r.t. this property, then $\exists$ complex of sheaves in m: $0 \to \mathcal{F} \to m \to (m_w^*)^\circ \to \mathcal{Y} \to 0$

+ this allows you to proceed by induction.