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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Emily Norton **Email/Phone:** nebenem@gmail.com

Speaker's Name: Miaofen Chen

Talk Title: Connected components of moduli spaces of p-divisible groups

Date: 8/29/14 **Time:** 3:30 am / pm (circle one)

List 6-12 key words for the talk: p-divisible group, isocrystal, Rapaport-Zink spaces, Dieudonné module, mixed characteristic affine Deligne-Lusztig variety, level structures

Please summarize the lecture in 5 or fewer sentences: Classifies p-divisible groups as isocrystals, surveys recent work by the author + collaborators on the connected components of such groups using combinatorial data (polytopes), and info this gives about the monodromy representation of a rigid analytic space, the R-Z space.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Connected components of moduli spaces of p-divisible groups

Miaofen Chen, 8/29/14

motivation: realize local Langlands, Jacquet-Langlands correspondences

1. p-divisible groups and their classification

p: prime S: scheme

Def A p-divisible group over S is an inductive system $(X_n)_{n\geq 1}$ of commutative finite locally free group schemes$/S$ s.t. $\forall n,k$:

$$\begin{tikzcd} 0 \arrow[r] & X_n \arrow[r] & X_{n+k} \arrow[r] \arrow[dr, "p^n"'] & X_k \arrow[r] \arrow[d] & 0 \\ & & & X_{n+k} & \end{tikzcd}$$ exact

ht $X := \log_p \operatorname{rk} X$, $S \to \mathbb{N}$ loc compact

Ex'ls
- $\mathbb{Q}_p/\mathbb{Z}_p = (p^{-n}\mathbb{Z}_p/\mathbb{Z}_p$, inclusion) ht $=1$
- E elliptic curve, $E[p^\infty]$ a p-div. gp. of ht 2
- A abelian variety, of dim d, then $A[p^\infty]$ is a p-div gp of height $2d$.

Want to classify p-div. gps. If S conn, p invertible on S, X/S p-div, X_n/S finite étale

$\bar{x} \in S \rightsquigarrow T_p : \{\text{p-div gps}/S\} \xrightarrow{\sim} \left\{\begin{array}{l}\text{finite free } \mathbb{Z}_p\text{-modules w/ cts}\\ \text{action of } \pi_1(S,\bar{x})\end{array}\right\}$

$$X \longmapsto T_p(X_{\bar{x}}) = \varprojlim_n (X_n)_{\bar{x}}$$

Now suppose S is a point, $S = \operatorname{Spec} k$, k: field of char p.
$W(k)$: ring of Witt vectors. $\sigma : W(k) \to W(k)$ Frobenius.

② <u>Def</u> A <u>crystal</u> over k is a pair (M, F) where

- M is a finite free $W(k)$-module
- $F: M \to M$ a σ-linear homomorphism s.t. $pM \subseteq FM \subseteq M$

Dieudonné: $\{p\text{-div gps}/k\} \xrightarrow{\sim} \{\text{crystals over } k\}$

$$X \longmapsto D(X) \quad \text{"Dieudonné module of } X\text{"}$$

- $\mathrm{ht}\, X = \mathrm{rk}_{W(k)} D(x)$
- $X_1 \xrightarrow[f]{} X_2$ isogeny $\Rightarrow D(X_1) \otimes \mathbb{Q} \xrightarrow[D(f)\otimes\mathbb{Q}]{} D(X_2) \otimes \mathbb{Q}$

<u>Def</u> An <u>isocrystal</u> over k is a pair (N, F) where

- N is finite-dim $W(k)\otimes\mathbb{Q}$-vector space
- $F: N \xrightarrow{\sim} N$ σ-linear isom

<u>Dieudonné–Manin</u> If $k = \bar{k}$, char $k = p$, then

1) the category of isocrystals$/k$ is semisimple with simple objects N_λ, $\lambda \in \mathbb{Q}$. If $\lambda = s/r$, $r > 0$, $(s, r) = 1$,

$$N_\lambda = \left(W(k)\otimes\mathbb{Q}^r, \begin{pmatrix} 0 & & p^s \\ 1 & \ddots & \\ 0 & \ddots & 1 \; 0 \end{pmatrix} \sigma \right)$$

If an isocrystal $N = \bigoplus_{i\in I} N_{\lambda_i}^{m_i}$, $m_i > 0$, then λ_i are called the "slopes" of the isocrystal N.

2) $\left\{ \begin{matrix} p\text{-div. gps}/k \\ \text{up to isogeny} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{isocrystals with} \\ \text{slopes in } [0,1] \end{matrix} \right\}$

$$X \longmapsto D(X) \otimes \mathbb{Q}$$

③ ## 2. Rapaport – Zink spaces

Fix $\mathbb{X}/\overline{\mathbb{F}}_p$, p-div gp.

$W = W(\overline{\mathbb{F}}_p) \circlearrowleft \sigma \qquad L = W \otimes \mathbb{Q} =$

Nil_W = cat of W-schemes where p is locally nilpotent

<u>Def</u> A <u>R-Z space</u> is a functor

$$\breve{\mathcal{M}} : Nil_W \to Sets$$
$$S \longmapsto \{X, \rho\}/\sim$$

where X p-div gp $/S$, $\rho : \mathbb{X} \times (S \bmod p) \to X \times_S (S \bmod p)$ a quasi-isogeny, i.e.

$\rho = p^{-n}\rho' \quad n \in \mathbb{Z}$, ρ' isogeny

<u>Rmk</u> $\breve{\mathcal{M}}$ only depends on the isogeny class of $\mathbb{X}$

(i.e. isocrystal $D(\mathbb{X}) \otimes \mathbb{Q}$)

$$D(\mathbb{X}) \otimes \mathbb{Q} = (L^n, b\sigma) \qquad b \in GL_n(L)$$

So $\breve{\mathcal{M}}$ depends on $b \in GL_n(L)$

In general we consider unramified R-Z-spaces of EL/PEL type:

EL: $\breve{\mathcal{M}} = \{(X, \rho, \iota)\}/\sim$

PEL: $\breve{\mathcal{M}} = \{(X, \rho, \iota, \lambda)\}/\sim$

$\iota : \mathcal{O}_F \to End(X)$

$\lambda : X \xrightarrow{\sim} X^\vee$

<u>R-Z</u> $\breve{\mathcal{M}}$ is representable by formal schemes / W

<u>Thm</u> (Kottwitz, Mantovan-Viehmann) If (G, b, μ) is HN-~~decomp~~ indec then $\exists M \subseteq G$ Levi with $b \in M(L)$, $\exists \mu' \in X_*(M)$ s.t. (M, b, μ') is HN-indec and the nat'l map $M(L)/M(\mathcal{O}_L) \to G(L)/G(\mathcal{O}_L) \rightsquigarrow X^M_{\mu'}(b) \xrightarrow{\sim} X^G_\mu(b)$

③ ④ $\check{M} = \check{M}(G, b, \mu)$ determined by datum (G, b, μ)

$G = \mathrm{Res}_{O_F/\mathbb{Z}_p} GL_n$, GU, GSp unramified red gp / $\mathbb{Z}_p$

$b \in G(L)$, $\mu \in X_*(G)$ minuscule cocharacter

$\check{M}^{an}$ = generic fiber of $\check{M}$ as rigid analytic space.

$\check{M}_K \subset G(\mathbb{Z}_p)$

$\downarrow$

$\check{M}^{an}$

<u>goal</u> $\pi_0(\check{M}_K \hat{\otimes} \mathbb{C}_p)$ $\qquad \mathbb{C}_p = \hat{\bar{L}}$

3. <u>Case $K = G(\mathbb{Z}_p)$</u>

$\check{M}_K = \check{M}^{an}$ $\qquad \pi_0(\check{M}^{an} \hat{\otimes} \mathbb{C}_p) = \pi_0(\check{M}^{an}) = \pi_0(\check{M}) = \pi_0(\check{M}_{red})$

$\check{M}(G, b, \mu) = \check{M}(\bar{\mathbb{F}}_p) = \{(X, \rho) \mid X \text{ p-div}/\bar{\mathbb{F}}_p,\ \rho : \mathbb{X} \to X \text{ q-i}\}/\sim$

$D(\mathbb{X}) \otimes \mathbb{Q} \cong (L^n, b\sigma)$

$= \{(\underset{\simeq M}{D(X)}, D(\rho)) \mid D(X) \text{ crystal},\ D(\rho) : D(\mathbb{X}) \otimes \mathbb{Q} \xrightarrow{\sim} D(X) \otimes \mathbb{Q}\}/\sim$

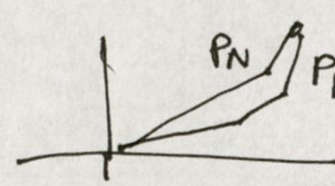
$= \{\underset{= g O_L^n}{M} \mid M \subset L^n \text{ lattice } pM \subseteq b\sigma M \subseteq M\}$ $\qquad g \in G(L)$

$= \{g \in G(L)/G(O_L) \mid g^{-1} b\sigma g \in G(O_L)\mu(p)G(O_L)\}$

(G, b, μ) "mixed characteristic affine DL variety" $\longrightarrow$ $X_\mu^G(b)$

P_N, P_H, polytopes $\qquad \pi_0(X_\mu^G(b)) = \pi_0(\check{M})$

If $X_\mu^G(b) \neq \emptyset$ then P_N lies above P_H:

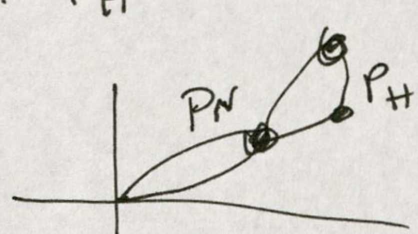


+ they have same endpts

<u>Def</u> (G, b, μ) is <u>HN-decomposable</u> if P_N has a break pt on P_H

If (G, b, μ) HN-indec then either $P_N(x) \neq P_H(x)$ except endpts $\to$ HN-irred, or $P_N = P_H$ straight $\to$ HN-degenerate

HN-dec:



Thm (C. - Kisin - Viehmann) ⑤

1) if (G,b,μ) HN-dec $\Rightarrow \pi_0(X^M_{\mu'}(b)) \cong \pi_0(X^G_\mu(b))$

2) (G,b,μ) HN-irred $\Rightarrow \pi_0(X^G_\mu(b)) = G^{ab}(\mathbb{Q}_p)/G^{ab}(\mathbb{Z}_p)$

3) (G,b,μ) HN-degen $\Rightarrow \pi_0(X^G_\mu(b)) = G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$

4. Case for general level structures K $\quad K \subseteq G(\mathbb{Z}_p)$

$$\check{\mathcal{M}}_K \xrightarrow{\text{finite étale}} \check{\mathcal{M}}^{an} \qquad\qquad X^{univ} \to \check{\mathcal{M}}^{an}$$

univ p-div gp, étale

$\bar{x} \in \check{\mathcal{M}}^{an}$

$\rightsquigarrow$ monodromy rep $\quad \rho_{\bar{x}} : \pi_1(\check{\mathcal{M}}^{an}, \bar{x}) \to Aut(T_p(X^{univ}_{\bar{x}})) = G(\mathbb{Z}_p)$

$\rightsquigarrow$ geom monodr rep

$$\rho^{geo}_{\bar{x}} : \pi_1(\check{\mathcal{M}}^{an} \hat{\otimes} \mathbb{C}_p, \bar{x}) \to G^{der}(\mathbb{Z}_p)$$

Thm (i) if (G,b,μ) HN-irred $\Rightarrow \rho^{geo}_{\bar{x}}$ surjective

(ii) (G,b,μ) HN-dec, (M,b,μ') HN-irred

$\Rightarrow \operatorname{Im} \rho^{geo}_{\bar{x}} = P^{der}(\mathbb{Z}_p)$ $\quad$ where P = parabolic of G, $M = \text{Levi}(P)$

Cor (1) (G,b,μ) HN-irred $\Rightarrow \pi_0(\check{\mathcal{M}}_K \hat{\otimes} \mathbb{C}_p) \cong G^{ab}(\mathbb{Q}_p)/\det K$

(2) (G,b,μ) HN-dec w/ (M,b,μ') HN-irred

$$\Rightarrow \pi_0(\check{\mathcal{M}}_K \hat{\otimes} \mathbb{C}_p) = P^{der}(\mathbb{Q}_p) \backslash G(\mathbb{Q}_p/K$$