MSRI – Langlands Program and the Fundamental Lemma

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MATHEMATICAL SCIENCES RESEARCH INSTITUTE

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December 1-December 5, 1986

MONDAY, December 1

NUMBER THEORY — 2:00 — MSRI Lec Hall
Jean-Marc Fontaine
"P-adic Periods IV"

TUESDAY, December 2

NUMBER THEORY — 2:05 — MSRI Lec Hall
G. Laumon
"Automorphic Sheaves"

NUMBER THEORY — 3:45 — MSRI Lec Hall
H. Hida
"Modules of Congruences and P-adic L-functions of Cusp Forms"

WEDNESDAY, December 3

No Lectures Scheduled.

THURSDAY, December 4

NUMBER THEORY — 2:00 — MSRI Lec Hall
Jim Milne
"Automorphic Vector Bundles"

- 8/28/86. Faltings, Minimal compactifications of moduli spaces
- 9/03/86. Gross, on Noam Elkies: Any elliptic curve E/Q has an infinite number of supersingular primes.
- 9/04/86. Zagier, A formula for $\zeta_K(2)$ and a conjecture for $\zeta_K(m)$, $m \geq 1$.
- 9/10/86. Vigneras. Stray course on local class field theory.
- 9/11/86. Casson, Counting representations of the fundamental group of a 3-manifold
- 9/16/86. Tunnell, On Fermat's conjecture
- 9/17/86. Gross, singular moduli
- 9/18/86. Eckmann. Euler characteristics of groups

- 9/23/86. Henniart, The numerical local Langlands conjecture for GL(n)
- 9/23/86. Ribet. Representations of G = $\mathrm{Gal}(\bar{Q}/Q)$ into GL(2,F), I
- 9/25/86. Vigneras. Heisenberg groups and dual reductive pairs
- 9/30/86. Ribet, II
- 10/1/86. de Shalit, Cyclotomic theory
- 10/1/86. Milne. Arithmetic of Automorphic Forms
- 10/2/86. Hales. The elliptic term of the trace formula and orbital integrals
- 10/6/86. Laumon. A global analysis of the nilpotent cone

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WORKSHOP ON GALOIS GROUPS OVER Q AND RELATED TOPICS

March 23-27, 1987

(All lectures will be held in the MSRI Lecture Hall, except on Friday when they will be in Room 10 of Evans Hall, Math Dept, Campus)

Monday, March 23

9:00- 9:30	Registration
9:30- 10:30	J.-P. Serre
10:30-10:45	Coffee
10:45-11:45	Y. Ihara
2:15- 3:15	P. Deligne
3:15- 3:45	Tea
3:45- 4:45	D. Blasius

Tuesday, March 24

9:10- 10:10	J.-P. Serre
10:10-10:45	Coffee
10:45-11:45	K. Ribet
2:15- 3:15	P. Deligne
3:15-3:45	Tea
3:45-4:45	B. Mazur

Wednesday, March 25

9:10 -10:10	B. Matzat
10:10-10:45	Coffee
10:45-11:45	W. Feit
2:15- 3:15	P. Deligne
3:15- 4:15	B. Mazur
4:15-...	Wine and cheese in the MSRI Lobby for conference participants MSRI members & staff.

Thursday, March 26

9:10-10:10	J.-P. Serre
10:10-10:45	Coffee
10:45-11:45	K. Ribet
2:15- 3:15	G. Anderson
3:15- 3:45	Tea
3:45- 4:45	K. Wingberg

Seminar talks during the 1986-87 year included: Kottwitz, Harder, Sarnak, Saito, Prasad, Ramakrishnan, Rubin, Michael Harris, Carayol, Rogawski, Soulé, Laumon, Katz, Hida, Silverberg, Zagier, Brylinski, Blasius, Greenberg, Coleman, Schneider, Bloch, Wintenberger, and Vignéras.

Outline of Lecture Series

1. Tuesday – Langlands program and the trace formula (characteristic zero) (Analytic representation theory)
2. Wednesday – The fundamental lemma (positive characteristic) (Geometric representation theory)
3. Thursday – Changing characteristics and motivic integration in the Langlands program (Logical representation theory)

Outline of Today's Talk (characteristic zero)

1. The trace formula – motivation and examples
2. Application: Tamagawa numbers
3. Local Langlands for $GL(n)$
4. Local Langlands for $Sp(2n)$
5. Langlands dual
6. Introduction to Endoscopy

Trace Formula for finite groups.

Let G be a finite group.

Let V = complex vector space of class functions:
$f(g^{-1}\gamma g) = f(g)$.

V has two canonical bases.

- The set of characteristic functions of conjugacy classes $C_\gamma = \{g^{-1}\gamma g : g \in G\}$.
- The set of irreducible characters of G: $\gamma \mapsto \operatorname{trace} \pi(\gamma)$, where π is an irreducible representation of G. Note that $\operatorname{tr}(A^{-1}BA) = \operatorname{tr}(B)$.

If $h \in V$ is any class function, it can be expanded in terms of these two bases:

$$\sum_C a_C(h) 1_C = \sum_\pi m_\pi(h) \operatorname{tr} \pi.$$

For example, if h is the character of a representation of G, then the left-hand side is an explicit character formula and the right hand side gives the multiplicities of irreducibles.

The identity can be transformed from an identity of functions into an identity of distributions. If $\phi : G \to \mathbb{C}$, there is a distribution (also denoted ϕ) such that

$$\phi(f) = \sum_G \phi(g) f(g)\, dg, \quad \forall f \in C_c^\infty(G).$$

Then the character identity becomes the **trace formula**

$$\sum_C a_C(h) 1_C = \sum_\pi m_\pi(h) \operatorname{tr} \pi.$$

Notationally, it the same formula as before, but now both sides are viewed as linear functionals on $C_c^\infty(G)$.

More explicitly, if 1_C is viewed as a distribution,

$$1_C(f) = \sum_G 1_C(g) f(g)\, dg = \sum_C f(c)\, dc = \sum_{G_\gamma \backslash G} f(g^{-1}\gamma g)\, dg$$

for $\gamma \in C$.

This is called an orbital integral. (Here orbit means an orbit under conjugation.)

Hence the trace formula states that a sum of orbital integrals equals a sum of irreducible characters.

Orbital integrals generalized to locally compact unimodular groups G by replacing the sum with an integral.

Poisson summation formula as trace formula

Let $\Lambda \subset \mathbb{R}^n$ be a lattice.

Let $\Lambda^* = \{y \in \mathbb{R}^n : x \cdot y \in \mathbb{Z}, \quad \forall x \in \Lambda\}$ be the dual lattice.

Let $\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot y}\, dx$. Note that this is just the distribution attached to the irreducible character $e^{-2\pi i(-\cdot y)}$ applied to f.

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a test function such that

- $\int_{\mathbb{R}^n} |f(x)| < \infty$
- $\sum_{x \in \Lambda} |f(x+u)|$ converges uniformly on compact sets.
- $\sum_{\Lambda^*} \hat{f}(y)$ converges absolutely.

Theorem 1 (Poisson summation)**.** *Under these conditions,*

$$\mathrm{vol}(\mathbb{R}^n/\Lambda) \sum_{x \in \Lambda} f(x) = \sum_{y \in \Lambda^*} \hat{f}(y).$$

The left-hand side is the **geometric side**. The right-hand side is the **spectral side**.

Note: Since Λ is abelian, a conjugacy class is just a singleton in Λ. The left-hand side is such a sum of orbital integrals of conjugacy classes in Λ. The right-hand side is just the sum of irreducible characters (viewed as distributions).

In general, a trace formula is a *nonabelian Poisson summation formula*.

Selberg trace formula for compact quotients

Let $\Gamma \subset G$ be a discrete co-compact subgroup of a locally compact unimodular topological group. (Say $\Lambda \subset \mathbb{R}^n$.)

G acts on $L^2(\Gamma\backslash G)$ by right translation. If R is this representation, we may view $\mathrm{tr}(R)$ as a distribution. The Selberg trace formula is

$$\begin{aligned} \mathrm{tr}(R)(f) &= \sum_{\gamma\in\{\Gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(g^{-1}\gamma g) \\ &= \sum m_\pi \, \mathrm{tr}(f), \end{aligned}$$

for $f \in C_c^\infty(G)$.

Let $\mathbb{A}_F$ be the ring of adeles of a number field F. Recall that this is a locally compact topological ring given as a restricted product

$$\mathbb{A}_F = \prod_v{}' F_v$$

where v runs over all places (equivalence classes of multiplicative norms on F), and F_v denotes the completion of F_v with respect to the metric given by the norm. For example, if $F = \mathbb{Q}$, one such place is the usual absolute value, with completion $\mathbb{R}$. The places are called archimedean or non-archimedean according to whether the the norm is unbounded or bounded on $\mathbb{Z} \subset F$.

It is known that F is discrete and co-compact in $\mathbb{A}_F$ under the diagonal embedding. In particular, we have a Poisson summation formula for $F \subset \mathbb{A}_F$ (Tate's thesis).

Arthur-Selberg trace formula (Selberg did rank 1)

But we obtain a much more interesting trace formula, if we choose G, a connected reductive group over a number field F.

$G(F)$ is discrete in $G(\mathbb{A}_F)$ and the quotient $G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ has finite volume. The Arthur-Selberg trace formula gives a similar expression for the trace of the representation of $G(\mathbb{A}_F)$ on the right of

$$L^2_{disc}(G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)).$$

The geometric side is a sum of orbital integrals (and other terms). The spectral side contains terms like the compact quotient case (and other terms).

Application: Tamagawa numbers

Let F be a number field. Now assume that G is semi-simple and simply connected. $G(F)\backslash G(\mathbb{A}_F)$ carries a canonical measure (coming from an invariant differential form of top degree on G). The volume is the Tamagawa number of G.

Theorem 2 (Langlands-Lai-Kottwitz)**.**

$$\mathrm{vol}(G(F)\backslash G(\mathbb{A}_F)) = 1.$$

Langlands proved the theorem for G split using the theory of Eisenstein series (Boulder conference). For a group over $\mathbb{Q}$, the proof comes down to calculating the volume of $G(\mathbb{Z})\backslash G(\mathbb{R})$.

Lai (1980) extended Langlands's proof to quasi-split groups.

Kottwitz (1988) used the trace formula to prove the theorem

By Lai's result, it is enough to show that the Tamagawa number for G is equal to the Tamagawa number for G^*, the quasi-split inner form.

The geometric side of the Arthur-Selberg trace formula contains the term

$$\mathrm{vol}(G(F)\backslash G(\mathbb{A}))f(1).$$

corresponding to the "orbital integral" of $1 \in G(F)$.

We have the trace formulas for G and G^*. Subtract one from the other:

$$\mathrm{geom}_G(f) - \mathrm{geom}_{G^*}(f) = \mathrm{spec}_G(f) - \mathrm{spec}_{G^*}(f).$$

By a careful choice of $f = \prod_v f_v$, Kottwitz is able to cancel all terms on the left except the term of interest:

$$(\mathrm{vol}(G(F)\backslash G(\mathbb{A})) - \mathrm{vol}(G^*(F)\backslash G^*(\mathbb{A})))f_u(1)$$

$$(\mathrm{vol}(G(F)\backslash G(\mathbb{A})) - \mathrm{vol}(G^*(F)\backslash G^*(\mathbb{A})))f_u(1)$$

as f_u runs over the spherical Hecke algebra at some quasi-split place u. The trace formula says that the sum over the spectral side is discrete, while local harmonic analysis says that the spectral formula for $f_u(1)$ is continuous. This forces the spectral side to vanish. Finally, this forces the term of interest to vanish.

Local Langlands for $GL(N)$.

Harris and Taylor have proved the local Langlands conjecture for $GL(N)$. We review this result.

Let F now be a non-archimedean local field of characteristic 0, with residue field k_F.

The Weil group W_F of F as an abstract group is the subgroup of $\mathrm{Gal}(\bar{F}/F)$ of elements whose image in $\mathrm{Gal}(\bar{k}_F/k_F)$ is a finite power of Frobenius. The map $W_F \to \mathrm{Gal}(\bar{F}/F)$ is continuous.

Let $L_F = W_F \times SU(2)$. Let $\Phi(N)$ be the set of equivalence classes of semisimple continuous representations of $L_F \to GL(N, \mathbb{C})$.

Let $\Pi(N)$ be the set of equivalence classes of irreducible admissible representations of $GL(N, F)$.

Theorem 3 (Harris-Taylor)**.** *For each* $N \geq 1$*, there exists a unique bijection* $\Phi(N) \leftrightarrow \Pi(N)$ *with properties.*

- $N = 1$ *is the bijection* $\Phi(1) = \Pi(1)$ *of local class field theory.*
- $\phi \otimes \chi \to \pi \otimes (\chi \circ \det)$.
- $\det \circ \phi =$ *central character of* π.
- $\phi^v = \pi^v$.
- $L(s, \phi_1 \times \phi_2) = L(s, \pi_1 \times \pi_2)$.
- $\epsilon(s, \phi_1 \times \phi_2, \psi_F) = \epsilon(s, \pi_1 \times \pi_2, \psi_F)$.

A similar bijection holds (by Langlands) for $GL(N)$ over archimedean fields. The local Langlands conjectures asks for a similar correspondence for any reductive group G.

Arthur has obtained a local Langlands correspondence for classical groups $Sp(2n, F)$ and $SO(N, F)$ (assuming work in progress on the stabilization of the twisted trace formula).

We describe his result for $G = Sp(2n, F)$.

Let Φ_{bdd} be the set of equivalence classes of semi-simple continuous representations $L_F \to SL(2n+1, \mathbb{C})$ whose image is relatively compact.

For each $\phi \in \Phi_{bdd}$, let $\mathfrak{S}_\phi = S_\phi / S_\phi^0$, where $S_\phi = \mathrm{Cent}(\mathrm{Im}\ \phi)$.

Let Π_{temp} be the set of equivalence classes of irreducible tempered representations of $Sp(2n, F)$.

Arthur's theorem [contingent of Walspurger's promised work] states the existence of an injective map

$$\Pi_{temp} \to \{(\phi, a) : \phi \in \Phi_{bdd}, \quad a \in \hat{\mathfrak{S}}_\phi\}.$$

This map is characterized by identities between characters of $Sp(2n, F)$ and twisted characters of $GL(2n+1, F)$.

Remarks: This (including a related statement for special orthogonal groups) is one of the main three theorems in Arthur's book. The other main theorems are global.

If θ is an automorphism of G (over F) and ω is a quasi-character of $G(F)$, there is a *twisted* orbital integral

$$\int_{\cdots} f(g^{-1}\gamma(\theta(g)))\omega(g)\,dg.$$

The automorphism θ can be inserted on the spectral side of the trace formula as well. An identity between twisted orbital integrals and the twisted spectral terms is a *twisted* trace formula.

The main tool in Arthur's book is the (stable) twisted trace formula. He must make comparisons among many trace formulas. The twisted trace formula of $GL(N)$ with outer automorphism θ, the standard trace formula for classical groups $Sp(2n, F)$ and $SO(N, F)$, and the twisted trace formula for $SO(2n, F)$ with outer automorphism.

Proof strategy (irredeemably simplified): each representation Φ_{bdd} gives $L_F \to SO(2n+1, \mathbb{C}) \to GL(2n+1, \mathbb{C})$, and by Harris and Taylor an irreducible admissible representation of $GL(2n+1, F)$. The difference of the twisted trace formula on $GL(2n+1, F)$ and the trace formula on $Sp(2n, F)$ is

$$\mathrm{geom}_{\theta, GL}(f) - \mathrm{geom}_{Sp}(f') = \mathrm{spec}_{\theta, GL}(f) - \mathrm{spec}_{Sp}(f').$$

Use the *twisted fundamental lemma* to relate the twisted orbital integrals of suitable f with the orbital integrals of f'. Cancel the geometric side. Deduce spectral relations.

Arthur "Endoscopic Classification of Representations: Orthogonal and Symplectic Groups" (590 pages, 2013).

Waldspurger (6 papers posted on the arXiv in 2014, with more to come on the stabilization of the twisted trace formula)

1. 18 jan, 137 pages, twisted endoscopy over a local field.
2. 28 jan, 105 pages, orbital integrals and endoscopy over a non-archimedean field (statements)
3. 12 feb, 95 pages, orbital integrals . . . (proofs)
4. 6 mar, just 35 pages, archimedean and spectral
5. 9 april, 91 pages, orbital integrals over $\mathbb{R}$
6. 9 june, 132 pages, geometric side of the twisted trace formula

Missing(?) spectral side of the stable twisted trace formula.

These 1200 pages (so far) merit an entry on Wiki's List of Long proofs. The list includes about 20 entries.

- 1880 Killing's classification of complex simple Lie algebras (180 pages)
- 1966 Harish-Chandra discrete series (150 pages)
- 1976 Langlands, Eisenstein series (337 pages)
- 1983 Hejhal, Selberg trace formula (1322 pages)
- 19XX Arthur, on the trace formula (several hundred pages)
- 2000, Lafforgue on the Langlands conjectures for $GL(n)$ over function fields (600 pages)

plus Weil Conjectures, Grothendieck, Hironaka, Almgren, 4-color theorem, Classification of Finite simple groups, etc.

Note that Arthur relates

- irreducible tempered representations of $Sp(2n, F)$,
- finite dimensional homomorphisms of L_F into $SO(2n+1, \mathbb{C})$.

This is an example of a general duality. Each reductive group G over a local field has a complex dual reductive group $\hat{G}$.

A big part of the Langlands program is to relate the representation theory of two reductive groups whenever their duals are related.

Analogy: Let V be a finite dimensional vector space over $\mathbb{R}$. For each $\lambda \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, we have a 1-dimensional representation of V: $v \mapsto e^{\lambda(v)}$.

The dual $\hat{G}$ of a reductive group G is a complex reductive group. An (inner or outer) form of G has the same dual $\hat{G}$.

- If (X^*, X_*, Φ, Φ^v) is the classifying data of G, then (X_*, X^*, Φ^v, Φ) is the data of $\hat{G}$.
- Duality exchanges short and long roots.
- Duals of groups with connected center are groups whose derived group is simply connected. (In particular, duals of adjoint groups are simply connected.)

Here are some examples.

- $GL(n) \leftrightarrow GL(n)$.
- $GSp(4) \leftrightarrow GSp(4)$.
- $SL(n) \leftrightarrow PGL(n)$.
- $SO(2n) \leftrightarrow SO(2n)$.
- $SO(2n+1) \leftrightarrow Sp(2n)$.

There is a more refined version of the the Langlands dual: ${}^L G = \hat{G} \rtimes W_F$ (or ${}^L G = \hat{G} \rtimes \mathrm{Gal}(\bar{F}/F)$). Every inner form of G has the same dual ${}^L G$, so we may take G to be quasi-split.

We have an action of $\mathrm{Gal}(\bar{F}/F)$ on (X^*, X_*, Φ, Φ^v) that permutes the set of positive simple roots. We have the same action on the dual data (X_*, X^*, Φ^v, Φ). By fixing a splitting of $\hat{G}$, we obtain an (algebraic action) of $\mathrm{Gal}(\bar{F}/F)$ as outer automorphisms of $\bar{G}$.

Example: $G = U(n)$ splits over a quadratic extension E/F. Over $\bar{F}$ (or E), G becomes isomorphic to $GL(n)$. Hence $\hat{G} = GL(n, \mathbb{C})$.

Let θ be the "transpose-inverse" outer automorphism of $\hat{G}$ of order 2 that acts non-trivially on the set of positive roots.

Let $\mathrm{Gal}(\bar{F}/F)$ act on $\hat{G}$ through its quotient $\mathrm{Gal}(E/F) \cong \langle\theta\rangle$, with semidirect product ${}^L G = \hat{G} \rtimes \mathrm{Gal}(\bar{F}/F)$.

The notation is not entirely standardized: $\hat{G} \rtimes \mathrm{Gal}(\bar{F}/F)$, $\hat{G} \rtimes W_F$, $\hat{G} \rtimes \mathrm{Gal}(E/F)$, etc.