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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im **Email/Phone:** mim2@illinois.edu

Speaker's Name: Pramod Achar

Talk Title: The Springer Correspondence

Date: 9/3/14 **Time:** 11:30 am / pm (circle one)

List 6-12 key words for the talk: The Springer resolutions, cohomology of the flag variety, skyscraper, constant, and perverse sheaves.

Please summarize the lecture in 5 or fewer sentences: Achar introduces classical results of the Springer resolution and the Springer correspondence. He discusses several classical results, including Borho-MacPherson's, which says if A is a certain Springer sheaf, then End(A) is isomorphic to the group algebra of W.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

The Springer Correspondence
Pramod Achar
Wed, Sept 3, 2014, 11:30 – 12:30 pm

<u>Plan</u>

char 0: Day 1: Ordinary — group
char 0: Day 2: Generalized — group / Lie algebra
Day 3: Modular — Lie algebra

G – conn reductive gp / $\mathbb{C}$
$\cup|$
B Borel subgp

$\mathcal{B} = G/B$ flag variety

$\mathcal{B}_g$ ($g \in G$) – Springer fiber

$$\mathcal{B}_g = \{ xB \in G/B : g \in xBx^{-1} \}$$

$\mathcal{U}$ = set of unipotent elmts in G.

<u>Observation</u> (1900)

$$\left\{ \begin{array}{c} \text{unipotent classes} \\ \text{in } GL_n \end{array} \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{c} \text{partitions of} \\ n \end{array} \right\} \longleftrightarrow \mathrm{Irr}(W) = \mathfrak{S}_n$$

$$C = \begin{bmatrix} \begin{smallmatrix} 1 & \ddots & 1 \end{smallmatrix} & & \\ & \begin{smallmatrix} 1 & \ddots & 1 \end{smallmatrix} & \\ & & \begin{smallmatrix} 1 & \ddots & 1 \end{smallmatrix} \end{bmatrix} \longmapsto \text{sizes of J. blocks}$$

In 1976, Springer discovered

$$W \circlearrowright H^\bullet(\mathcal{B}_u, \mathbb{C}), \quad u \in \mathcal{U}$$

Moreover, among the $H^{top}(\mathcal{B}_u, \mathbb{C})$, each irred. W-rep occurs for a unique u up to conj.

Get $\mathrm{Irr}(W) \longrightarrow \left\{ \begin{array}{c} \text{unip.} \\ \text{classes} \end{array} \right\}$

LATER

Enrich this side with info. about $A_G(u) = C_G(u) / C_G^\circ(u)$.

Warm-up: special case

$\mathcal{B}_e = \mathcal{B}$

id elmt.

Trick

1.) $G/T \longrightarrow G/B$ induces isom. in cohomology

$$H^\bullet(G/B) \xrightarrow{\simeq} H^\bullet(G/T)$$

2.) $W = N_G(T)/T$ acts on G/T by multn. on the right.

What is the action?

Let V = refl. rep. of W.

$S(V)$ = sym. algebra on V.

$$\text{Coinv.}(W) = S(V) \big/ \left(S(V)^W_+ \right)$$

homog. W-invts of pos degree.

<u>Thm</u> (Borel, 1953)

$$H^*(\mathcal{B}) \simeq \text{Coinv.}(W)$$

<u>Other Springer fibers</u>

$$\tilde{G} := \{ (g, xB) \in G \times \mathcal{B} : x^{-1} g x \in B \}$$

$$\begin{array}{c} \tilde{G} \\ \pi \downarrow \\ G \end{array} \qquad \text{By defn, } \pi^{-1}(g) = \mathcal{B}_g.$$

$$\tilde{\mathcal{U}} = \pi^{-1}(\mathcal{U}).$$

Main diagram

$$\begin{tikzcd}
\mathcal{B}_u \arrow[r, hook] \arrow[d] & \tilde{\mathcal{U}} \arrow[r, hook] \arrow[d, "\text{"Springer resolution"}"] & \tilde{G} \arrow[d, "\pi"] & \tilde{G}_{rs} \arrow[l, hook'] \arrow[d, "\text{"Galois side"}"] \\
\{u\} \arrow[r, hook] & \mathcal{U} \arrow[r, hook] & G & G_{rs} \arrow[l, hook', "j"', "\text{open}"]
\end{tikzcd}$$

$G_{rs} \supseteq h$

Automorphic side

Step 0 Note:

$$R\pi_* \underline{\mathbb{C}}\big|_{\{u\}} = H^\bullet(\mathcal{B}_u)$$

Plan. 1.) Get W to act on $R\pi_* \underline{\mathbb{C}}$.
2.) Decompose $R\pi_* \underline{\mathbb{C}}\big|_{\mathcal{U}}$.

Lemma ① Let $G_{rs} = \{\text{reg ss elmts in } G\}$
$\tilde{G}_{rs} = \pi^{-1}(G_{rs})$.

Then $\tilde{G}_{rs} = \{(g, xT) \in G \times G/T : x^{-1}gx \in T\}$.

Lemma ② $W \curvearrowright \tilde{G}_{rs}$.

$\pi|_{\tilde{G}_{rs}} : \tilde{G}_{rs} \to G_{rs}$ is a Galois cov. map w/ covering group W.

Cor ③. Let $\mathcal{L} = (R\pi_* \underline{\mathbb{C}})|_{G_{rs}}$

Then $\mathcal{L}$ is the loc. sys. corresp. to reg. rep. of W.

In particular, $\text{End}(\mathcal{L}) = \mathbb{C}[W]$.

Thm ④ a) $R\pi_* \underline{\mathbb{C}}[\dim G]$ is perverse

$\simeq j_{!*}(\mathcal{L}[\dim G])$

b.) $R\pi_* \underline{\mathbb{C}}[\dim \mathcal{U}]|_{\mathcal{U}}$ is a perv. sh. on $\mathcal{U}$.

Proof (sketch)

Perv. sh. are characterized by some cohom.-vanishing bounds. Prove those bounds by bounding $\dim \mathcal{B}_g$.

Cor. of 4(a.) + ③

$$\text{End}(\underbrace{R\pi_*\underline{\mathbb{C}}}_{\text{lives on } G}) = \mathbb{C}[W].$$

Restrict to $\mathcal{U}$: get a map

(*) $\mathbb{C}[W] \longrightarrow \text{End}(R\pi_* \underline{\mathbb{C}}|_{\mathcal{U}})$.

Let $A := R\pi_* \underline{\mathbb{C}}|_{\mathcal{U}}[\dim \mathcal{U}]$ ← the Springer sheaf

Thm 5 (Borho – MacPherson)

(*) is an isom. So
$\mathrm{End}(A) \simeq \mathbb{C}[W]$.

(Pf. Restrict further to $\mathcal{B}_e$, use warm-up.).

Decomposition Thm $\Rightarrow$
A is a semisimple perv. sh. i.e.,

$$A \simeq \bigoplus_{(C,\mathcal{E})} IC(C,\mathcal{E}) \otimes V_{C,\mathcal{E}}$$

unipotent class

equiv. irredu loc. sys. OR rep. of $A_G(u)$.

"multiplicity vec. space"

Thm 5 $\Rightarrow$

$$A \simeq \bigoplus_{V \in \mathrm{Irr}(W)} \left(\text{simple obj.}\right) \otimes V$$

distinct

Match terms:
get $\boxed{\mathrm{Irr}(W) \underset{\text{inj. map}}{\hookrightarrow} \{(C,\mathcal{E})\}}$

Springer Correspondence

OR $\boxed{\mathrm{Irr}(W) \hookrightarrow \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{U}))}$

Examples

1.) GL_n : All $A_G(u)$'s are triv.
So no nontriv. loc. sys.
$$\{(C,\mathcal{E})\} = \{(C,\underline{\mathbb{C}})\}.$$

Springer correspondence is a bij.

2.) SL_2 $\quad u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A_G(u) \simeq \mathbb{Z}/2$

$\left(\text{class of } J, \begin{array}{c}\text{nontriv}\\ \text{loc.}\\ \text{sys.}\end{array}\right)$ is not in the image of the Springer corresp.

Problem Explain the missing pairs.

Setup for Thurs :
fix a parabolic subgp $P = LV$

$$i_P : P \hookrightarrow G, \quad q_P : P \twoheadrightarrow L.$$

Two functors :
$$\mathrm{res}^G_{L\subseteq P} : D^b_G(G) \to D^b_L(L)$$
$$\mathrm{res}^G_{L\subseteq P} = Rq_{P!}\, i_P^*$$

Rmk. If $H \curvearrowright X$, $H \subseteq G$, then
$$D^b_G(G \times^H X) = D^b_H(X)$$
$G \times X/(gh, x) \sim (g, hx)$.

$$G\times^H(-) : D^b_H(X) \to D^b_G(G\times^H X)$$

$$\mathrm{ind}^G_{L\subseteq P} : D^b_L(L) \to D^b_G(G)$$

$$\| $$

$$R\pi_* \underbrace{G\times^P(q^! p(-))}_{\text{sheaf on } G\times^P P \xrightarrow{\pi} G}$$

<u>Exercise</u>. $A = \mathrm{ind}^G_{T\subseteq B}\ \underline{\mathbb{C}}_{\{e\}}$

$\uparrow$

skyscraper on $e\in T$.