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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im **Email/Phone:** mim2@illinois.edu

Speaker's Name: Victor Ginzburg

Talk Title: Geometry of Quiver Varieties

Date: 9/2/14 **Time:** 2:00 am (pm) (circle one)

List 6-12 key words for the talk: Quiver varieties, representation space moduli stack of ab. category of fin. dim'l repr. of a fixed quiver, Donaldson-Thomas invariants

Please summarize the lecture in 5 or fewer sentences: Ginzburg discusses quivers and quiver varieties. He then shows that the representations of a quiver have homological dimension 1, i.e., such ~~and~~ stack of objects is smooth. Ginzburg introduces the notion of Hamiltonian reduction for quivers and Donaldson-Thomas invariants.

arXiv: 0905.0686

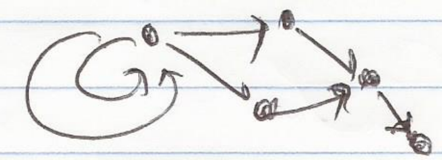
CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
    - **Overhead**: Obtain a copy or use the originals and scan them
    - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - **Handouts**: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Geometry of Quiver Varieties
Victor Ginzburg
Tues Sept 2, 2014, 2–3pm

Quiver
I = vertex set
oriented arrows



A rep ρ of a quiver Q is an assignment of a fin. dim. v. space V_i $\forall i \in I$.

A linear map $\hat{e} : V_i \to V_j$ $\forall$ arrow $e : i \to j$.

$\{\dim V_i\} = \dim \rho \in \mathbb{Z}^I_{\geq 0}$

vector space $\operatorname{Rep}_\alpha Q$ = set of all reps of $\dim = \alpha \in \mathbb{Z}^I_{\geq 0}$

$V_i = \mathbb{C}^{\alpha_i}$

$G_\alpha = \prod_{i \in I} GL_{\alpha_i}(\mathbb{C}) = \prod_{i \in I} GL(V_i) \curvearrowright \operatorname{Rep}_\alpha Q$

$\operatorname{Rep}_\alpha Q / G_\alpha$ = stack = moduli stack of ab. cat. (objects) of fin. dim'l reps of a fixed quiver Q

Fact Stack of objects of a cat. $\mathcal{C}$ is **smooth** iff hom. dim $\mathcal{C} \leq 1$.

i.e., $\mathrm{Ext}^i_{\mathcal{C}}(M,N) = 0 \quad \forall i > 1 \quad \forall M, N \in \mathcal{C}$.

Thm $\mathcal{C}$ has hom. dim. 1 only in the following cases:

1.) $\mathcal{C}$ = reps of a Q
2.) Coh. sheaves on a smooth curve
3.) Lots of exceptional cases

k field, $kI = k^I$, k-basis 1_i (semi simple reps)

$E = \bigoplus_{e \text{ edge}} ke$, a kI bimodule

$$1_i e 1_j = \begin{cases} e & \text{if source of } e = i,\ \text{target of } e = j, \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Path alg. of } Q = \underbrace{T_{kI}}_{\text{tensor algebra}} E = P$$

Observation rep of Q = P-module

Lemma $\forall$ P-mod M, N, we have $\mathrm{Ext}^j_P(M,N) = 0$ $\forall j > 1$.

Standard explicit resolution:

$$0 \longrightarrow P \otimes_{kI} E \otimes_{kI} P \longrightarrow P \otimes_{kI} P \xrightarrow{\text{mult.}} P \longrightarrow 0$$

free, proj.

$$p \otimes e \otimes p' \mapsto pe \otimes p' - p \otimes ep'$$

For $P = T_{kI} E$, this is an exact sequence.

Resoln $\otimes_P M$:

$$0 \longrightarrow P \otimes_{kI} E \otimes_{kI} M \to P \otimes_{kI} M \to M$$

$\mathrm{Hom}(*, N)$:

$$0 \to \mathrm{Hom}_P(M,N) \to \mathrm{Hom}_{kI}(M,N) \to \mathrm{Hom}_{kI}(E \otimes M, N)$$

$$\to \mathrm{Ext}^1_P(M,N) \to 0$$

__Euler form__ : $\mathbb{Z}^I \times \mathbb{Z}^I \longrightarrow \mathbb{Z}$

$\alpha, \beta \in \mathbb{Z}^I$, $\langle \alpha, \beta \rangle = \sum_{i \in I} \alpha_i \beta_i - \sum \alpha_i \dim E_{ij} \beta_j$

$E_{ij} = \#\{i \to j\}$

exact sequence:

$\dim \mathrm{Hom}(M,N) - \dim \mathrm{Ext}^1(M,N)$

$= \dim M \cdot \dim N - \sum \dim M_i \cdot \dim E_{ij} \cdot \dim N_j$

Ringel form:

$\chi(\mathrm{Ext}^\bullet) = \langle \dim M, \dim N \rangle$.

<u>DT invariants</u> (after Kontsevich–Soibelman).

Q quiver, P = path alg. of Q

W cyclic potential $= P/[P,P]$.

cyclic quotient of a path algebra

Fix $\alpha \in \mathbb{Z}^I_{\geq 0}$.

$W_\alpha : \mathrm{Rep}_\alpha Q \longrightarrow k$

$\rho \longmapsto \mathrm{Tr}\, \rho(W_\alpha)$

$G_\alpha = \prod GL_{\alpha_i}$ inv. fn.

descends to $\mathrm{Rep}_\alpha Q / G_\alpha \longrightarrow k$

There is a constructible sheaf of vanishing cycles $\Phi(W_\alpha)$ on $W_\alpha^{-1}(0)$.

$$\chi\left(H^\bullet(W_\alpha^{-1}(0), \phi_\alpha(W_\alpha))\right) = \widetilde{DT}_\alpha$$

Formal generating series:

$$\mathrm{Log}\left(\sum_\alpha q^\alpha \cdot \widetilde{DT}_\alpha\right) = DT(Q, W)$$

$$H^\bullet(W_\alpha^{-1}(0), \phi(W_\alpha)) =$$

$$= H^\bullet(\Omega_{Rep_\alpha Q}, d_{DR} + dW_\alpha)$$

"<u>Fact</u>" $\forall$ CY3 category,

$$\rightsquigarrow (Q, W) \rightsquigarrow DT(Q, W)$$

Reminder on symplectic geometry

$T^*X \xrightarrow{p} X$, $G \curvearrowright X$ action of an alg. gp on X

$$\begin{array}{c} T^*X \\ \downarrow \mu \\ \mathfrak{g}^* \end{array} \qquad \mathfrak{g} = \mathrm{Lie}\, G$$

action of a at x

$$T_x^*X \ni \phi \longrightarrow [\mu(\phi) : \underset{\mathfrak{g}}{\underset{\cap}{a}} \longrightarrow \langle \phi, \underset{T_xX}{\underset{\cap}{a_x}} \rangle]$$

Hamiltonian reduction

$G \curvearrowright X \rightsquigarrow G \curvearrowright T^*X$

$\mathrm{Ad}\, G \curvearrowright \mathfrak{g} \rightsquigarrow \mathrm{Ad}^* G \curvearrowright \mathfrak{g}^*$

μ is G-equivariant

$\forall h \in \mathfrak{g}^*$, $\mu^{-1}(h)$ is a G_h-stable subvariety of T^*X,

$\mu^{-1}(h)/G_h$:= Hamiltonian reduction.

<u>Concrete case</u> : $X = \mathrm{Rep}_\alpha Q$

$G = G_\alpha$ acts on $\mathrm{Rep}_\alpha Q$

$\mu : T^* \mathrm{Rep}_\alpha Q \longrightarrow \mathfrak{g}_\alpha^*$, $\mathfrak{g}_\alpha = \mathrm{Lie}\, G_\alpha$

A "quiver variety" $\mu^{-1}(h)/G_h$

Relation to Kac's conjecture (1381), proved 2013 (HLRV).

Ground field $\mathbb{F}_q$

$$\#\ \text{absolutely indecomp reps in } \mathrm{Rep}_\alpha(Q, \mathbb{F}_q) = \underbrace{a_\alpha(q)}_{\text{poly. in } q \text{ with nonnegative coefficients.}}$$

<u>Thm</u> $ai_\alpha(q) = \mathrm{Tr}_{Fr}\, H^\bullet(\mu^{-1}(\lambda)/G_\lambda, \overline{\mathbb{Q}}_\ell)$

for λ sufficiently generic.

An analogue of DT-invariants / $\mathbb{F}_q$.

Fix $\psi : (\mathbb{F}_q, +) \to \overline{\mathbb{Q}}_\ell^\times$, a mult char,

$$DT(Q, W, \mathbb{F}_q)_\alpha = \sum_{\substack{\text{abs. ind.}\\ \rho \in \text{reps. of } Q\\ \dim \alpha}} \psi(W_\alpha(\rho))$$