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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im Email/Phone: mim2@illinois.edu

Speaker's Name: Olivier Schiffmann

Talk Title: Quivers, curves, Kac polynomials and the number of stable Higgs bundles

Date: 9/2/14 Time: 3:30 am / pm (circle one)

List 6-12 key words for the talk: quivers, Kac polynomials, indecomposable vector bundles, stable Higgs bundles.

Please summarize the lecture in 5 or fewer sentences: Schiffmann replaces the category of representations of a quiver by the category of coherent sheaves on a smooth proj curve, explaining a "global" analog of a certain conjecture by Kac. As an application, Schiffmann gives a formula for the number of stable Higgs bundles over a sm proj. curve over a finite field.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
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- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
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Quivers, curves, Kac polynomials
and the number of stable Higgs bundles
Olivier Schiffmann
Tues, Sept 2, 2014, 3:30–4:30pm

Position in categorical Langlands correspondence
X sm proj. curve $/\mathbb{C}$
$G, G^\vee$ Langlands dual groups
$(G = G^\vee = GL(r))$

Rough form of cat. Langlands corresp.
$\mathcal{O}\text{-mod}(\mathrm{Loc}_{G^\vee} X)$ "$\simeq$" $D\text{-mod}(\mathrm{Bun}_G X)$

E irredn loc. system
$\rightsquigarrow \quad \mathcal{O}_E \quad \rightsquigarrow \quad \mathrm{Aut}_E$

What about more general torsion sheaves on $\mathrm{Loc}_{G^\vee} X$?

Most singular pt; E = trivial local system
$\mathcal{O}\text{-mod}_{triv}(\mathrm{Loc}_{G^\vee} X) \rightsquigarrow$ a certain category of D-modules (perverse sheaves?)

$\exists$ parabolic induction functors on both sides

$L \leftarrow P \rightarrow G$ parabolic subgp, Levi factor

$$L^\vee \leftarrow P^\vee \rightarrow G^\vee$$

induce maps

$$\mathrm{Loc}_{L^\vee} \leftarrow \mathrm{Loc}_{P^\vee} \rightarrow \mathrm{Loc}_{G^\vee}$$

$$\mathrm{Bun}_L \leftarrow \mathrm{Bun}_P \rightarrow \mathrm{Bun}_G$$

"inducing" functors

$$\mathcal{O}\text{-mod}(\mathrm{Loc}_{L^\vee} X) \rightleftarrows \mathcal{O}\text{-mod}(\mathrm{Loc}_{G^\vee} X)$$

$$D\text{-mod}(\mathrm{Bun}_L X) \rightleftarrows D\text{-mod}(\mathrm{Bun}_G X)$$

More manageable categories
subcategories generated by
$\mathcal{O}_{triv}(\mathrm{Loc}_{T^\vee} X)$ + $\mathbb{C}_{\mathrm{Bun}_T}$

$G = GL_r$

$$\begin{tikzcd} \mathrm{Bun}_B X \arrow[r, hook, "open"'] \arrow[d, "\pi"'] & \mathrm{Coh}_B X \arrow[d, "\pi'"] \\ \mathrm{Bun}_G X \arrow[r, hook, "open"'] & \mathrm{Coh}_G X \end{tikzcd}$$

π not proper
π' proper

$\rightsquigarrow$ $\mathrm{Eis}_G X$ = subcategory generated by $\pi'_!(\mathbb{C}_{\mathrm{Coh}_B X})$.

Have functors

$$\mathcal{E}is_{GL_{n+m}} \rightleftarrows \mathcal{E}is_{GL_n} \times \mathcal{E}is_{GL_m}$$

induces a structure of a Hopf algebra on

$$\bigoplus_n K_0(\mathcal{E}is_{GL_n})$$

Hope / Observation: these categories $\mathcal{E}is_{GL_n} X$

+ their K_0
have a rich + rigid structure

$\rightsquigarrow$ ∞ - infinite dim quantum groups, DAHA
Combinatorics
Knot theory
Rep theory,

Analogy with quivers

$\vec{Q}$ quiver
$\underline{d}$ dim vector
$Bun_{GL_n} X \rightsquigarrow Rep_{\underline{d}} \vec{Q}$

$\mathcal{E}is_{GL_n}$ $\rightsquigarrow$ $\mathcal{Q}_{\underline{d}}$ cat. of perverse sheaves introduced by Lusztig
$\rightsquigarrow$ canonical bases.

$$\bigoplus_n K_0(\mathcal{E}is_{GL_n}) \rightsquigarrow \bigoplus_{\underline{d}} K_0(\mathcal{Q}_{\underline{d}})$$

$$\simeq U_q^+(\mathfrak{g}_{\vec{Q}})$$

Lusztig Ringel

Kac-Moody alg ass to $\vec{Q}$.

Thm (Varagnolo-Vasserot) Fix $\vec{Q}$, $\underline{d}$.

$$\mathrm{Ext}^\circ\left(\bigoplus_{\mathbb{P}} \mathbb{P}, \bigoplus_{\mathbb{P}} \mathbb{P}\right)$$

where $\mathbb{P}$ runs among simple objs in $\mathcal{Q}_{\underline{d}}$.
is isom to KLR-alg $A_{\vec{Q},\underline{d}}$.
(fundamental tool to categorification).

Aim: use "combinatorics" of $\bigoplus_r K_0(\mathcal{E}is_{GL_r} X)$
to compute

* Cohomology of moduli space of stable Higgs bundles $\mathrm{Higgs}^{st} X$ for $X/\mathbb{C}$

* $\#(\mathrm{Higgs}^{st} X)$ for $X/\mathbb{F}_q$.

§ 1. Facts about vector bundles + Higgs bundles on curves

X sm proj. curve $/k$, $k \in \{\mathbb{C}, \mathbb{F}_q, \overline{\mathbb{F}}_q\}$

$Coh(X)$ - abelian cat. of gl dim $=1$
For $\mathcal{F} \in Coh(X)$,
$$cl(\mathcal{F}) = (rk(\mathcal{F}), \deg(\mathcal{F})) \in \mathbb{Z}^2$$

$$\langle \mathcal{F}, \mathcal{G} \rangle := hom(\mathcal{F}, \mathcal{G}) - ext(\mathcal{F}, \mathcal{G}) \underset{R\text{-}R}{=} (1-g) r_{\mathcal{F}} r_{\mathcal{G}} + (r_{\mathcal{F}} d_{\mathcal{G}} - r_{\mathcal{G}} d_{\mathcal{F}})$$

$Bun_{r,d}$ - moduli stack of v. bdles on X of $cl = (r,d)$
Smooth stack, locally of finite type of dim $= -\langle (r,d), (r,d) \rangle = (g-1) r^2$

* Cohomology of $Bun_{r,d}$

Let $\mathcal{E}$ be the universal bdle on $X \times Bun_{r,d}$, i.e.,

$$\mathcal{E}|_{X \times \{E\}} = E$$

$$c_i(\mathcal{E}) \in H^*(X \times Bun_{r,d}) = H^*(X) \otimes H^*(Bun_{r,d})$$

Set $c_i(\mathcal{E}) = [X] \otimes a_i + \sum_1^{2g} \gamma_j \otimes b_i^j$
$+ [pt] \otimes f_i$

$a_i, b_i^j, f_i \in H^*(\mathrm{Bun}_{r,d})$

<u>Thm</u>. (Atiyah–Bott, Harder–Narasimhan)

$$H^*(\mathrm{Bun}_{r,d}) = \mathbb{C}\langle a_i, b_i^j, f_i \rangle_{i=1,\dots,r}$$

* <u>Point count</u>:

$k = \mathbb{F}_q$.

$\mathrm{Bun}_{r,d}(\mathbb{F}_q)$ is a groupoid

$$\mathrm{vol}(\mathrm{Bun}_{r,d}(\mathbb{F}_q)) = \sum_{v \in \mathrm{Bun}_{r,d}} \frac{1}{\# \mathrm{Aut}(v)}.$$

Let $\zeta_X(z) = \dfrac{\prod_{i=1}^{2g} (1 - z\alpha_i)}{(1-z)(1-qz)}$, $\alpha_1, \dots, \alpha_{2g}$ Frob eig. on $H^1(\bar{X}, \overline{\mathbb{Q}}_\ell)$ "Weil numbers of X"

<u>Thm</u> (Harder, Siegel, Langlands)

$$\mathrm{vol}(\mathrm{Bun}_{r,d}) = \frac{q^{(g-1)(r^2-1)}}{q-1} |\mathrm{Jac}\, X(\mathbb{F}_q)| \prod_{i=2}^{r} \zeta_X(q^{-i})$$

* Relation between H^* & vol

Thm (?, Heinloth, Schmitt)
$X/\mathbb{F}_q$

$H^*(\mathrm{Bun}_{r,d})$ is pure.

* Semistable bundles

Def $\mu(\mathcal{F}) = \dfrac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \in \mathbb{Q}$

$\mathcal{F}$ is $\frac{1}{2}$-stable if $\forall\, \mathcal{G} \subseteq \mathcal{F}$,
$\mu(\mathcal{G}) \leq \mu(\mathcal{F})$.

$$\begin{array}{ccc} \mathrm{Bun}_{r,d}^{1/2\,st} & \underset{\text{open}}{\subseteq} & \mathrm{Bun}_{r,d} \\ \big\downarrow \text{coarse mod space} & & \\ \mathcal{M}_{r,d} & & \end{array}$$

$\mathcal{M}_{r,d} \longleftarrow$ moduli space of $\frac{1}{2}$ stable v. bundles (projective).

Harder-Narasimhan filtration

Fact. $\forall\, \mathcal{V} \in \mathrm{Bun}_{r,d}$ $\exists!$ filtration
$\mathcal{V}_s \subseteq \mathcal{V}_{s-1} \subseteq \cdots \subseteq \mathcal{V}_1 = \mathcal{V}$
s.t. $\mathcal{V}_i / \mathcal{V}_{i+1}$ is semi-stable and
$\mu(\mathcal{V}_i/\mathcal{V}_{i+1}) < \mu(\mathcal{V}_{i+1}/\mathcal{V}_{i+2})$

Def. type of $v = \left(\mathrm{sl}(v_1/v_2), \ldots, \mathrm{sl}(v_s) \right)$

$\rightsquigarrow$ partition of $\mathrm{Bun}_{r,d} = \coprod_{\underline{\alpha}} HN_{\underline{\alpha}}$

$$HN_{\underline{\alpha}} \longrightarrow \mathrm{Bun}^{\frac{1}{2}st}_{\alpha_1} \times \ldots \times \mathrm{Bun}^{\frac{1}{2}st}_{\alpha_s}$$

|| stack vector bundle
"affine fibration"

$\rightsquigarrow$ allows you to recursively compute
$\mathrm{vol}\left(\mathrm{Bun}^{\frac{1}{2}st}_{r,d} \right)$

$H^*\left(\mathrm{Bun}^{\frac{1}{2}st}_{r,d} \right)$

$\rightarrow$ ref. Lectures by J. Heinloth
"Moduli stack of v.b. on curves".

<u>Higgs bundles</u>

$\mathrm{Higgs}_{r,d} = T^* \mathrm{Bun}_{r,d}$ singular stack

$T_E \mathrm{Bun}_{r,d} = \mathrm{Ext}^1(E, E)$

$$T_E^* \mathrm{Bun}_{r,d} = \mathrm{Ext}^1(E,E)^* \underset{\text{Serre}}{\simeq} \mathrm{Hom}(E, E \otimes \Omega)$$

$$\mathrm{Higgs}_{r,d} = \{ (\mathcal{F}, \theta) : \mathcal{F} \in \mathrm{Bun}_{r,d},\ \theta \in \mathrm{Hom}(\mathcal{F}, \mathcal{F} \otimes \Omega) \}$$

$\gcd(r,d)=1$

$\exists$ $\mathrm{Higgs}^{st}_{r,d}$ moduli space, smooth

$\mu : \mathrm{Higgs}^{st}_{r,d} \longrightarrow \mathbb{A}$ Hitchin map, Lagrangian Fibration

<u>Compute</u>: $H^*(\mathrm{Higgs}^{st}_{r,d})$
$\#(\mathrm{Higgs}^{st}_{r,d}(\mathbb{F}_q))$

Fix $g \geq 0$.

$T_g = \{(z_1, \dots, z_{2g}) \in \mathbb{G}_m^{2g} : z_{2i-1} z_{2i} = z_{2j-1} z_{2j} \ \forall i,j\}$
$\subseteq GSp(2g)$

$W_g \cong S_g \ltimes S_2^{\,g}$

$\forall \mathbb{F}_q$, $\forall X/\mathbb{F}_q$ of genus g,

$\sigma_X \in T_g / W_g$
$\quad \| $
$\{\text{Weil nhbds}\}$

<u>Thm</u> Fix g, fix r,d, $\gcd(r,d)=1$
i) $\exists !$ explicit poly $A_{g,r,d} \in \mathbb{N}[-z_1, \dots, z_{2g}]^{S_g}$
s.t. $\forall \mathbb{F}_q$; $\forall X$ of genus g, $\mathbb{Q}[T_g]^{W_g}$

$$A_{g,r,d}(\sigma_X) = q^{-r^2(g-1)+1} \# \mathrm{Higgs}^{st}_{r,d}(\mathbb{F}_q)$$

ii.) — / $\mathbb{C}$

iii) $A_{g,r,d}(\sigma_X) = \#$ abs. indec.
v. b. on X of rk r
& deg d.