MSRI – the Fundamental Lemma

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Outline of Lecture Series

1. Tuesday – Langlands program and the trace formula (characteristic zero) (Analytic representation theory)
2. Wednesday – The fundamental lemma (positive characteristic) (Geometric representation theory)
3. Thursday – Changing characteristics and motivic integration in the Langlands program (Logical representation theory)

Almost the same thing works, if we convert functions to distributions

If G is a reductive group/ local field, a trace formula is a linear relation among

- characters as distributions

$$f \mapsto \mathrm{trace} \int_G f(g)\pi(g)\,dg, \quad f \in C_c^\infty(G),$$

- conjugacy classes as distributions:

$$f \mapsto \mathbf{O}(\gamma, f) = \int_{I_\gamma \backslash G} f(g^{-1}\gamma g)\,dg, \quad f \in C_c^\infty(G),$$

where I_γ is the centralizer of $\gamma \in G$.

Selberg trace formula for compact quotients

Let $\Gamma \subset G$ be a discrete co-compact subgroup of a locally compact unimodular topological group. (Say $\Lambda \subset \mathbb{R}^n$.)

G acts on $L^2(\Gamma\backslash G)$ by right translation. If R is this representation, we may view $\mathrm{tr}(R)$ as a distribution. The Selberg trace formula is

$$\begin{aligned}\mathrm{tr}(R)(f) &= \sum_{\gamma\in\{\Gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(g^{-1}\gamma g) \\ &= \sum m_\pi \,\mathrm{tr}\,\pi(f),\end{aligned}$$

for $f \in C_c^\infty(G)$.

Arthur-Selberg trace formula (Selberg did rank 1)

But we obtain a much more interesting trace formula, if we choose G, a connected reductive group over a number field F.

$G(F)$ is discrete in $G(\mathbb{A}_F)$ and the quotient $G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ has finite volume. The Arthur-Selberg trace formula gives a similar expression for the trace of the representation of $G(\mathbb{A}_F)$ on the right of

$$L^2_{disc}(G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)).$$

The geometric side is a sum of orbital integrals (and other terms). The spectral side contains terms like the compact quotient case (and other terms).

Proof strategy (irredeemably simplified): each representation Φ_{bdd} gives $L_F \to SO(2n+1,\mathbb{C}) \to GL(2n+1,\mathbb{C})$, and by Harris and Taylor an irreducible admissible representation of $GL(2n+1,F)$. The difference of the twisted trace formula on $GL(2n+1,F)$ and the trace formula on $Sp(2n,F)$ is

$$\mathrm{geom}_{\theta,GL}(f) - \mathrm{geom}_{Sp}(f') = \mathrm{spec}_{\theta,GL}(f) - \mathrm{spec}_{Sp}(f').$$

Use the *twisted fundamental lemma* to relate the twisted orbital integrals of suitable f with the orbital integrals of f'. Cancel the geometric side. Deduce spectral relations.

A big part of the Langlands program is to relate the representation theory of two reductive groups whenever their duals are related.

The dual $\hat{G}$ of a reductive group G is a complex reductive group. An (inner or outer) form of G has the same dual $\hat{G}$.

- If (X^*, X_*, Φ, Φ^v) is the classifying data of G, then (X_*, X^*, Φ^v, Φ) is the data of $\hat{G}$.
- Duality exchanges short and long roots.
- Duals of groups with connected center are groups whose derived group is simply connected. (In particular, duals of adjoint groups are simply connected.)

Here are some examples.

- $GL(n) \leftrightarrow GL(n)$.
- $GSp(4) \leftrightarrow GSp(4)$.
- $SL(n) \leftrightarrow PGL(n)$.
- $SO(2n) \leftrightarrow SO(2n)$.
- $SO(2n+1) \leftrightarrow Sp(2n)$.

There is a more refined version of the the Langlands dual: ${}^LG = \hat{G} \rtimes W_F$ (or ${}^LG = \hat{G} \rtimes \mathrm{Gal}(\bar{F}/F)$). Every inner form of G has the same dual ${}^LG$, so we may take G to be quasi-split.

We have an action of $\mathrm{Gal}(\bar{F}/F)$ on (X^*, X_*, Φ, Φ^v) that permutes the set of positive simple roots. We have the same action on the dual data (X_*, X^*, Φ^v, Φ). By fixing a splitting of $\hat{G}$, we obtain an (algebraic action) of $\mathrm{Gal}(\bar{F}/F)$ as outer automorphisms of $\bar{G}$.

TOPICS: perverse sheaves, purity, middle extension, local systems, decomposition theorem, affine Springer fibers, spectral curves, Hitchin fibration, cameral covers, Higgs pairs, completely integrable systems, torsors, G-bundles, gerbs, stacks (Artin and Deligne-Mumford), endoscopic groups, Langlands dual, **fundamental lemma**, stable conjugacy, Tate-Nakayama duality, mass (groupoid cardinality), groupoids, orbital integrals, weighted orbital integrals, faithfully flat descent, weak abelian fibrations, Poincaré duality, Pontryagin product, spectral sequences, Weil restriction, Arthur-Selberg trace formula, stabilization, Hecke algebras, Galois cohomology, Grothendieck Lefschetz trace formula, Kostant section, transfer factors, reductive groups, polarized abelian varieties, Picard stack, Chevalley's theorem, Satake transform

Credits:

Langlands, Shelstad
Waldspurger
Kottwitz, Goresky, MacPherson
Laumon, Ngô

many many others: Labesse, Chaudouard, Arthur, Kazhdan,. . .

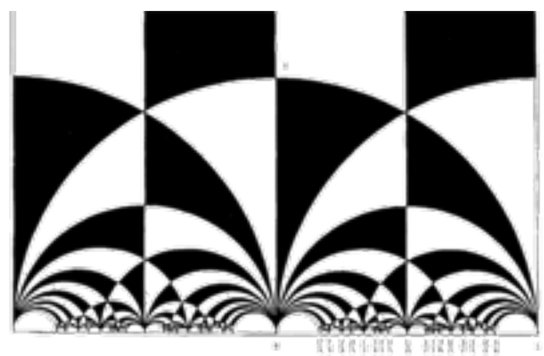
The example of 2 by 2 matrices

$n \in \mathbb{N}$, $n \geq 2$,
$V_{n,+}$ = space of holomorphic functions f on the upper half plane $\mathfrak{h}$ such that

$$\int_{\mathfrak{h}} |f|^2 y^{n-2} dx\, dy < \infty.$$

$SL_2(\mathbb{R})$ acts on $V_{n,+}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) = (-bz + d)^{-n} f(\frac{az - c}{-bz + d}).$$



$V_{n,-}$ = anti-holomorphic discrete series.
$\Theta_{n,\pm}$ characters of $V_{n,\pm}$.

The basic example that leads to the FL

The characters are equal: $\Theta_{n,+}(g) = \Theta_{n,-}(g)$, except when g is conjugate to a rotation

$$\gamma = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

When g is conjugate to γ, a remarkable character identity holds:

$$\Theta_{n,-}(\gamma) - \Theta_{n,+}(\gamma) = \frac{e^{i(n-1)\theta} + e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}}.$$

This has the general form

$$\text{alternating sum} = \frac{\text{character on smaller group}}{\text{transfer factor}}$$

Problem: find and prove in full natural generality.

A six-fold generalization

From character identity to the FL:

$$\Theta_{n,-}(\gamma) - \Theta_{n,+}(\gamma) = \frac{e^{i(n-1)\theta} + e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}}$$

alternating sum = denominator × sum on smaller group

$$\mathbf{O}_\kappa(\nu(a)) = q^{r_v(a)}\mathbf{SO}_H(a)$$

- Replace SL_2 with reductive group G.
- Replace $\mathbb{R}$ with nonarchimedean local field F_v.
- Replace characters with orbital integrals.
- Replace signs $\pm$ on LHS with roots of unity $\langle\kappa,\gamma\rangle$.
- Replace rotation group with H (endoscopic).
- Replace denominator with transfer factor $q^{\cdots}$.

The six-fold generalization.

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Clockwise and counterclockwise are not conjugate.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

in $SL_2(\mathbb{R})$ are conjugate by the complex matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$,

but they are not conjugate in the group $SL_2(\mathbb{R})$ when $\theta \notin \mathbb{Z}\pi$.

Let G be a reductive group defined over a field F with algebraic closure $\bar{F}$.

Definition 1. *An element $\gamma' \in G(F)$ is said to be stably conjugate to a given regular semisimple element $\gamma \in G(F)$ if γ' is conjugate to γ in the group $G(\bar{F})$.*

$$\Theta_{n,-}(\gamma) - \Theta_{n,+}(\gamma) = \frac{e^{i(n-1)\theta} + e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}}.$$

A finite abelian group

Let I_γ be the centralizer of an element $\gamma \in G(F)$. Write

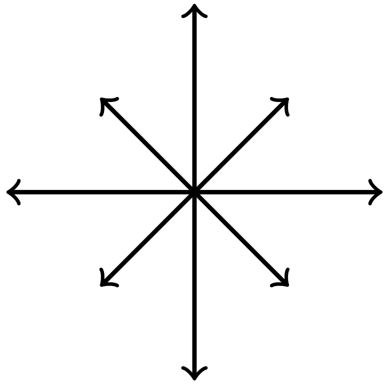
$$\gamma' = g^{-1}\gamma g,$$

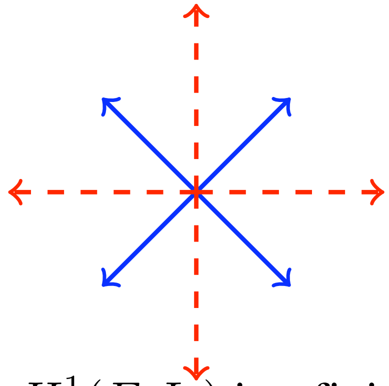
for $g \in G(\bar{F})$. With $\sigma \in \mathrm{Gal}(\bar{F}/F)$, we have

$$g\,\sigma(g)^{-1} \in H^1(F, I_\gamma)$$

- The class does not depend on the choice of g.
- It is the trivial class when γ' is conjugate to γ.
- When F is a local field and γ is regular semisimple, $A = H^1(F, I_\gamma)$ is a finite abelian group.

The effect of oscillation on root systems





$A = H^1(F, I_\gamma)$ is a finite abelian group. Every function $A \to \mathbb{C}$ has a Fourier expansion as a linear combination of characters $\kappa : A \to \mathbb{C}$.

Roughly, the Fourier mode of κ (for given I_γ and G) produces oscillations that cause some of the roots of G to "cancel" and the others to become more "pronounced". The mode of κ on the group G should be related to the dominant mode on H.

$$\Theta_{n,-}(\gamma) - \Theta_{n,+}(\gamma) = \frac{e^{i(n-1)\theta} + e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}}.$$

Recall the basic philosophy: if duals are related then the representation theory should be related.

The simplest way in which duals are related occurs when one dual is a subgroup of another dual group, given by the connected centralizer of a semisimple element:
$\hat{H} = C_{\hat{G}}(s)^0 \subset \hat{G}$.

When this happens we say that the smaller group H is an *endoscopic group* of the larger group G. (This will be made more precise later.)

Endoscopic group

The smaller group H, formed from the *pronounced* subset of the roots of G, is called an endoscopic group.
There is a surjection

$$\hat{T}^{\Gamma} \to H^1(F, I_\gamma)^*$$

Definition 2 (endoscopic group)**.** *Let F be a local field. The endoscopic group H associated with (G, I_γ, κ) is defined as follows.*

$$\hat{H} = \hat{I}_\kappa^0 \subset \hat{G}.$$

Pick quasi-split form H by forcing isomorphic Cartan subgroups I_H of H with I_γ in G.

Characteristic function of a matrix

Take ring of polynomial functions $k[x_{ij}]$ on the vector space of n by n matrices.

It has a subring $k[x_{ij}]^{GL(n)}$ of polynomial functions f such that $f(g^{-1}\gamma g) = f(\gamma)$ for all $g \in GL(n)$. This subring is generated by the coefficients c_i of the characteristic polynomial

$$p(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_0 \quad (\text{:}$$

of a matrix $\gamma \in \mathfrak{g} = \mathfrak{gl}(n)$. The morphism $\chi : \mathfrak{g} \to \mathfrak{c}$ is the "characteristic map" that sends γ to $(c_{n-1}, \ldots, c_0)$.

Chevalley generalization of characteristic polynomials

For simplicity, we move from conjugacy in the group G to conjugacy in the Lie algebra $\mathfrak{g}$.

Let G be a split reductive group over a field k and let $\mathfrak{g}$ be its Lie algebra, with split Cartan subalgebra $\mathfrak{t}$ and Weyl group W. Assume that the characteristic of k is sufficiently large. The group G acts on $\mathfrak{g}$ by the adjoint action. By Chevalley,

$$k[\mathfrak{g}]^G = k[\mathfrak{t}]^W.$$

Set $\mathfrak{c} = \mathrm{Spec}\,(k[\mathfrak{t}]^W)$ and $\chi : \mathfrak{g} \to \mathfrak{c}$.

Two regular elements γ, γ' are stably conjugate $\leftrightarrow$ $\chi(\gamma) = \chi(\gamma')$.

The kappa orbital integral generalizes the LHS of the character sum

k finite field

F_v nonarchimedean local field containing k

O_v integers of F_v

$a \in \mathfrak{c}(F_v)$ regular semisimple

J_a centralizer $I_{\gamma_0}, \ \gamma_0 = \epsilon(a)$

$\kappa : H^1(F_v, J_a) \to \mathbb{C}^\times$

$\langle \kappa, \gamma \rangle$ the pairing of of κ with the cohomological invariant of γ. A κ-orbital integral:

$$\mathbf{O}_\kappa(a) = \sum_{\chi\gamma = a} \int_{I_\gamma \backslash G(F_v)} \langle \kappa, \gamma \rangle 1_{\mathfrak{g}(\mathcal{O}_v)}(\mathrm{Ad}\, g^{-1}(\gamma))dg,$$

$1_{\mathfrak{g}(\mathcal{O}_v)}$ the characteristic function of $\mathfrak{g}(\mathcal{O}_v)$

Haar measure dg, and quotient by dt.

$$\Theta_{n,-}(\gamma) - \Theta_{n,+}(\gamma) = \frac{e^{i(n-1)\theta} + e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}}.$$

Now for the RHS of the identity

κ determines H over $\mathcal{O}_v$

Add subscripts for H: $\mathfrak{c}_H$, etc.

$\nu : \mathfrak{c}_H = \mathfrak{t}/W_H \to \mathfrak{t}/W = \mathfrak{c}$.

If κ is trivial, write $\mathbf{SO}$ for $\mathbf{O}_\kappa$.

Theorem 4 (fundamental lemma (FL), Ngô (2010))**.** *Assume that the characteristic of F_v is greater than twice the Coxeter number of G. For all regular semisimple elements $a \in \mathfrak{c}_H(\mathcal{O}_v)$ whose image $\nu(a)$ in $\mathfrak{c}$ is also regular semisimple, the κ orbital integral of $\nu(a)$ in G is equal to the stable orbital integral of a in H, up to a power of q:*

$$\mathbf{O}_\kappa(\nu(a)) = q^{r_v(a)}\mathbf{SO}_H(a), \quad \textit{where} \ \ r_v(a) = \deg_v(a^*\mathfrak{R}).$$

The FL is the natural generalization of the character identity:

$$\Theta_{n,-}(\gamma) - \Theta_{n,+}(\gamma) = \frac{e^{i(n-1)\theta} + e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}}$$

The FL has been simplified (by Waldspurger and others) from its original statement by Langlands (1980).

The fundamental lemma is the key identity between orbital integrals that is needed for the comparison of the trace formulas for G and H. The fundamental lemma makes it possible to extract relationships between representations of H and representations of G.

Coset geometry

The integrand

$$1_{\mathfrak{g}(\mathcal{O}_v)}(\mathrm{Ad}\, g^{-1}(\gamma))dg$$

is invariant under $g \mapsto g' \in g\, G(\mathcal{O}_v)$. So the orbital integral counts cosets, weighted by roots of unity $\langle \kappa, \gamma \rangle$, meeting the support of the integrand.

Coset counting approaches to the FL failed (for good reason).

Kazhdan-Lusztig (1988)

$$\mathcal{M}_v(a, \bar{k}) = \{g \in G(\bar{F}_v)/G(\bar{\mathcal{O}}_v) \mid \mathrm{Ad}\, g^{-1}\gamma \in \mathfrak{g}(\bar{\mathcal{O}}_v)\},$$

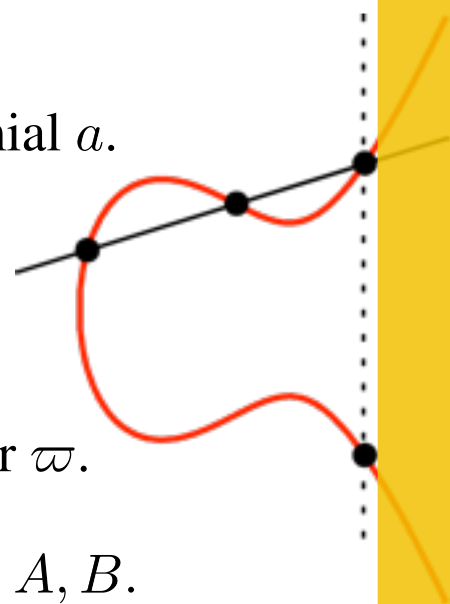
where $\gamma = \epsilon(a)$, is the set of $\bar{k}$-points of an ind-scheme called the *affine Springer fiber*.

Let’s do a calculation!

Orbital integrals count points on curves

Try $\mathfrak{so}(5)$ = vector space of 5 by 5 skew-symmetric matrices.
$a \in \mathfrak{c}(F_v)$ characteristic polynomial
$0, \pm t_1, \pm t_2$ the eigenvalues of a matrix
$\gamma = \epsilon(a) \in \mathfrak{so}(5) \subset \mathfrak{gl}(5)$ with characteristic polynomial a.
Assume for some odd $r \in \mathbb{N}$,

$$|\alpha(\gamma)| = q^{-r/2}, \quad \forall \text{ roots } \alpha.$$



$$E_a : y^2 = (1 - x^2\tau_1)(1 - x^2\tau_2), \quad \tau_i = t_i^2/\varpi^r, \text{ unifzr } \varpi.$$

$$\mathbf{SO}(a, f) = A(q) + B(q)\,\mathrm{card}(E_a(k)), \text{ rational fn. } A, B.$$

FL as isogeny

There is a variant of the FL that only involves $\kappa = 1$. In this case the Lie algebra of the endoscopic group is $\mathfrak{sp}(4)$.

The corresponding calculation of orbital integrals in $\mathfrak{sp}(4)$ gives a different elliptic curve E'_a, but otherwise identical to the formula for $\mathfrak{so}(5)$.

E_a and E'_a have different j-invariants.

FL $\leftrightarrow$ isogeny between E_a and E'_a.

The FL seems to involve the geometry of abelian varieties?!

Similar calculations of orbital integrals for $\mathfrak{so}(2n+1)$ yield $y^2 = (1 - x^2\tau_1)\cdots(1 - x^2\tau_n)$, hyperelliptic curves.

Main work on the FL before Ngo

Goresky, Kottwitz, and MacPherson made an extensive investigation of affine Springer fibers and conjectured that their cohomology groups are pure. Assuming this conjecture, they prove the FL for elements whose centralizer is an unramified Cartan subgroup. They prove the purity result in particular cases by constructing pavings of the affine Springer fibers.

Laumon has made a systematic investigation of the affine Springer fibers for unitary groups. Ngô joined the effort, and together they succeeded in giving a complete proof of the FL for unitary groups.

Limits to earlier approaches

Ngô encountered two major obstacles in trying to generalize this earlier work to an arbitrary reductive group.

(1) **Torus Actions:** These approaches calculate the equivariant cohomology by passing to a fixed point set in $\mathcal{M}_v(a)$ under a torus action. In general, a nontrivial torus action simply doesn't exist.

(2) **Purity:** The second serious obstacle: the purity conjecture itself. Deligne's work suggests proving purity in a family of varieties, rather than individually. Ngô investigated families varying over a base curve X, moving us from local geometry of F_v to the global geometry of X. He found that the

Hitchin fibration

is the global analogue of affine Springer fibers.

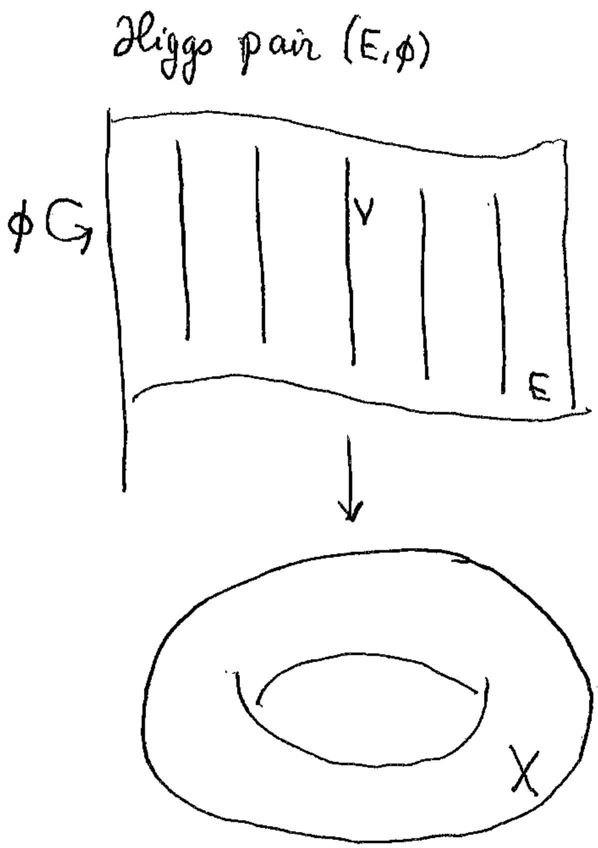
Schiffmann described the stack of Higgs bundles and the Hitchin map in yesterday's lecture.

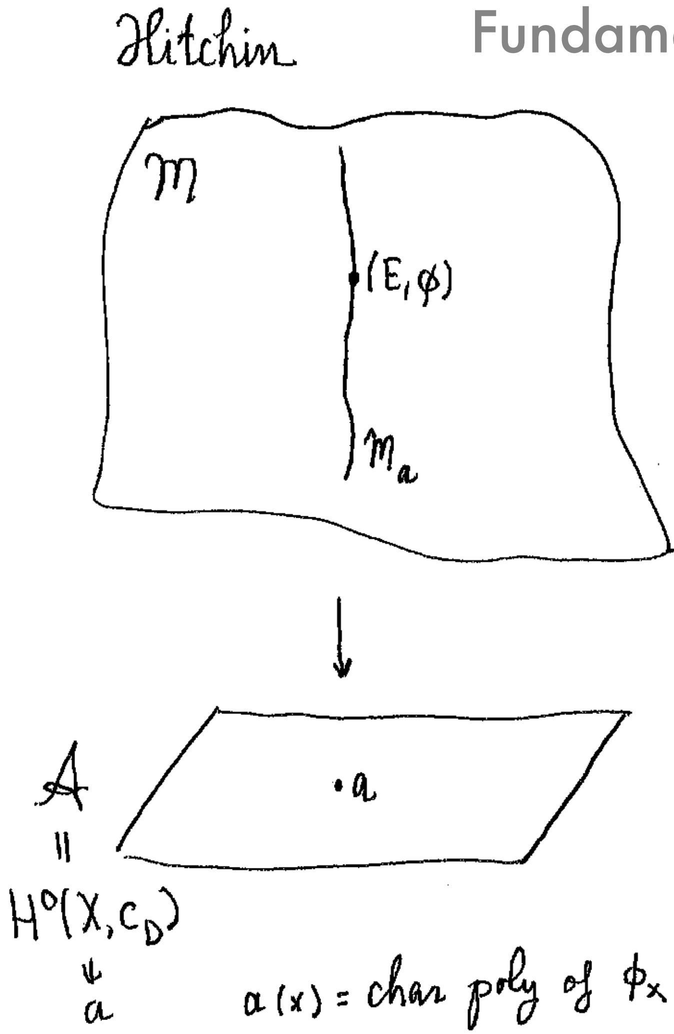
Châu's proof of the FL is based on the Hitchin map (for a general reductive group G) with mild changes. No stability condition is imposed on bundles.

A Higgs pair is a pair (E,ϕ) where E is a G-torsor on X and ϕ is a section of $\mathrm{ad}(E) \otimes_{O_X} D$, where D is a line bundle and $\mathrm{ad}(E)$ is the vector bundle on X coming from the adjoint representation on the Lie algebra of G.

Note: D is not the same as yesterday. Because of this change, in our context, the moduli stack of Higgs bundles is not symplectic, and there is no completely integrable system. The Hitchin map is not Lagrangian.

Hitchin fibration for GL(n)





The starting point of NBC's proof of the FL is the following theorem:

Theorem 1 (Ngô)**.** *There is an explicit test function f_D, depending on the line bundle D, such that for every anisotropic element $a \in \mathcal{A}^{an}$, the sum of the orbital integrals with characteristic polynomial a in the trace formula for f_D equals the number of Higgs pairs in the Hitchin fibration over a, counted with multiplicity.*

The proof is based on Weil's description of vector bundles on a curve in terms of the cosets of a compact open subgroup of $G(\mathbb{A})$. Orbital integrals have a similar coset description.

The **stabilization** of the geometric side of the trace formula consists of two manipulations of the trace formula.

- Rearrange the terms according to the Fourier expansion of characters κ of $H^1(F, I_\gamma)$.
- For non-trivial κ, use the fundamental lemma (and related identities) to replace the κ-terms with stable terms on the corresponding endoscopic group H.

With the interpretation of orbital integrals as counting points on Hitchin fibers, we can try to manipulate the Hitchin fibers in parallel with the manipulation of the trace formula.

Hitchin fibration

X = smooth projective curve of genus g over k.

G reductive group over X

$\mathfrak{g}, \mathfrak{c}, \ldots$ over X

D = line bundle on X.

E vector bundle on X with fiber V

φ a section of $\mathrm{end}(V) \otimes D$. The Hitchin fibration $\mathcal{M}_{G,X,D}$ for $G = GL(n) = GL(V)$ is given by the groupoid[a] of pairs (E, φ).

[a] A groupoid is a category in which all morphisms are invertible

Hitchin for general G

Generally, $\mathcal{M} = \mathcal{M}_{G,X,D}$ is the stack that assigns to a k-scheme S the groupoid of pairs

$$(E, \varphi),$$

$E = G$-torsor over $X \times S$
$\varphi =$ section of $\mathrm{Ad}(E) \otimes D = \mathrm{Ad}(E)_D$.

The fiber of $\mathrm{Ad}(E)$ is $\mathfrak{g}$, and the morphism $\chi : \mathfrak{g} \to \mathfrak{c}$, fiber by fiber, gives a morphism $f(\phi) \in \mathfrak{c}_D(X \times S)$.

$$f : \mathcal{M} \to \mathcal{A} = H^0(X, \mathfrak{c}_D).$$

The fiber of f over $a \in \mathcal{A}$ is $\mathcal{M}_a$.

Key ideas

Key ideas[a] of the Proof of the FL

- Hitchin fibration is the global analogue of the affine Springer fiber.
- The Hitchin fibration carries a large symmetry group $\mathcal{P}(J_a)$ that is well-suited for the FL and endoscopic groups.
- Continuity arguments can be used.
- The purity conjecture can be replaced globally (with pure perverse sheaves).

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[a] For endoscopy, use the same X and D for both G and its endoscopic group H.

Local-Global correspondence

affine Springer fiber	Hitchin fibration
local	global
G-torsors on $X_v \hat{\times} S \ldots$	G-torsors on $X \times S \ldots$
local orbital integrals	adelic orbital integrals

There is an equivalence of categories at the level of $\bar{k}$-points:

$$\prod_{v \in \bar{X} \setminus U} \mathcal{M}_v(a) \wedge \cdots \qquad \text{and} \qquad \mathcal{M}_a$$

(U = good places for a)

Many of us believed that a global argument could not work, because the FL is precisely what is needed to start using global arguments. We said global arguments are bound to be circular.

"I also had occasion to listen to lectures of Ngô . . . In particular, I had to attach for myself some meaning to the notion of stack and algebraic stack. It was a revelation. I discovered that I had been thinking for decades of orbital integrals in an incorrect way. I had separated the local from a global part." Langlands, Shaw Prize.

Second key idea: symmetry

Symmetries of the Hitchin fibration

For $G = GL(n)$, $P(J_a) = \mathrm{Pic}(Y_a)$ acts on $\mathcal{M}_a$ where Y_a is an n-fold cover of X (depending on $a = (c_{n-1}, \ldots, c_0)$):

$$t^n + c_{n-1}(v)t^{n-1} + \cdots + c_0(v) = 0, \quad v \in X.$$

(a sufficiently generic)

Symmetry in general

I_γ (centralizer), for $\gamma \in \mathfrak{g}$, depends only on $a = \chi(\gamma)$. $J_a = I_\gamma$, as we vary $a \in \mathfrak{c}$, define a smooth group scheme J over $\mathfrak{c}$. For each $a : S \to \mathcal{A}$, there is a groupoid $\mathcal{P}(J_a, S)$ whose objects are

$$J_a\text{-torsors on } X \times S.$$

Moreover, $\mathcal{P}(J_a, S)$ acts on $\mathcal{M}_a(S)$.

As the S-point a varies, we obtain a *Picard stack* $\mathcal{P}$ acting fiberwise on the Hitchin fibration $\mathcal{M}$.

Symmetry

Symmetry and Endoscopy

In general, it is hard to pass geometric information between G and H.

$$\text{No morphism } \ H \to G.$$

However, for $a \in \mathcal{A}_H$:

$$J_{\nu(a)} \to J_{H,a}, \quad \nu : \mathcal{A}_H \to \mathcal{A}.$$

is an isomorphism over a nonempty open set of X. So their Picard groups are also directly related.

Strategy: view $\mathcal{M}_{\nu(a)}$ in terms of $\mathcal{P}(J_{\nu(a)})$. Express FL (insofar as possible) in terms of Picard.

Third Key Idea: Continuity

Continuity

The complexity of an orbital integral is measured by the dimension of its affine Springer fiber.

$\dim \mathcal{M}_v(a) \sim \deg_v(a^*\mathfrak{D}) \to \infty, \quad \mathfrak{D} =$ discriminant divisor.

Approximate a with a' that satisfies transversality: $\deg_w(a'^*\mathfrak{D}) \leq 1$ for all $w \in X$.

The justification of continuity is the deepest part of Ngô's work.

Fourth key idea: BBDG decomposition

Locally the orbital integral is the number of k-points on an affine Springer fiber.

Globally, the orbital integral is the number of k-points on a Hitchin fibration.

This number is computed as a perverse sheaf on $\mathcal{A}^{\mathrm{ani}}$.

The geometric form of the fundamental lemma takes the form of an equality of (the semisimplifications of) two perverse sheaves

$$\nu^* L_\kappa \stackrel{?}{=} L_{H,st}$$

$$L_\kappa = {}^pH^n(\tilde{f}_*^{\mathrm{ani}}\bar{\mathbb{Q}}_\ell)_\kappa, \quad L_{H,st} = {}^pH^{n+2r}(\tilde{f}_{H,*}^{\mathrm{ani}}\bar{\mathbb{Q}}_\ell)_{st}(-r)$$

on a common base space $\nu(\mathcal{A}_H^{\mathrm{ani}})$.

BBDG decomposition

On an open subset U of this base space (consisting of transverse elements a'), the FL is a relatively easy direct calculation. Hence the (ss. of the) perverse sheaves are equal on U. However, two perverse sheaves can be equal on a dense open U without being equal.

By BBDG decomposition theorem (over $\bar{k}$):

- If Z is closed irreducible,
- If two simple perverse sheaves have support Z.
- If they are equal on a dense open subset U.
- Then they are equal on Z.

The perverse sheaves are almost as good as simple ones.

The BBDG theorem is a continuity result that provides the infrastructure for Ngô's proof of the FL by continuity. Even though in the FL the perverse sheaves are not simple, Ngô proves that every (good) geometrically simple factor has full support. This support theorem is the technical heart of the proof.

The proof of the support theorem breaks into two parts:

- Every support Z of every simple geometric factor also appears as the support of some factor in the ordinary cohomology of top degree.
- Supports in top degree ordinary cohomology are as large as possible.

Proof of the support theorem.

[A]n exposition [of the FL] genuinely accessible not alone to someone of my generation, but to mathematicians of all ages eager to contribute to the arithmetic theory of automorphic representations, would be, perhaps, ... close to 700 pages. – Langlands

What's the use of the FL?

"Lemma" is a misleading name for the Fundamental Lemma because it went decades without a proof, and its depth goes far beyond what would ordinarily be called a lemma.

Yet the name FL is apt both because it is fundamental and because it is expected to be used widely as an intermediate result in many proofs.

We mention some major theorems that have been proved that contain the FL as an intermediate result. In each case, the FL appears to be an unavoidable ingredient.

Arthur-Selberg trace formula

The Arthur-Selberg trace formula

G reductive group over a number field F

$$\sum_{\gamma \in G(F)/\sim} c_\gamma \mathbf{O}(\gamma, f) + \cdots = \sum_{\pi} m(\pi)\, \mathrm{trace}\, \pi(f) + \cdots$$

LHS = sum of adelic orbital integrals

RHS = sum over discrete series automorphic representations

Use the FL to rewrite the LHS:

$$\sum_{H} \sum_{\gamma \in H(F)/\simeq} c'_\gamma \mathbf{SO}(\gamma, f^H) + \cdots$$

The endoscopic group are carried along in G's penumbra wherever the trace formula is used. All applications of the FL come through the trace formula.

From elementary cases of the FL comes...

Classical uses of early cases of the FL.

- cyclic base change $\rightarrow$ certain cases of the Artin conjecture $\rightarrow$ Fermat.
- the calculation of the Hasse-Weil zeta function of some simple Shimura varieties.
- FL$/SL(n)$ $\rightarrow$ local automorphic induction $\rightarrow$ local Langlands conjectures for $GL(n)$.

General Sato-Tate laws

Fundamental Lemma

An theorem that uses the FL from a recent paper by Barnet-Lamb, Geraghty, Harris and Taylor

Let n_p be the number of ways a prime p can be expressed as a sum of twelve squares:

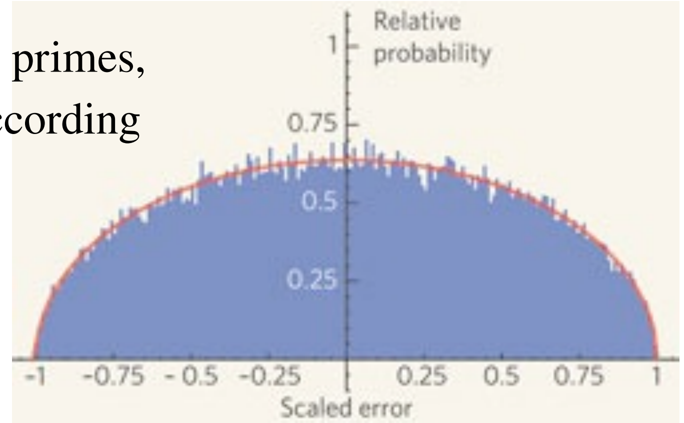
$$n_p = \text{card}\,\{(a_1, \ldots, a_{12}) \in \mathbb{Z}^{12} \mid p = a_1^2 + \cdots + a_{12}^2\}.$$

Then the real number

$$t_p = \frac{n_p - 8(p^5+1)}{32p^{5/2}}$$

belongs to the interval $[-1, 1]$, and as p runs over all primes, the numbers t_p are distributed within that interval according to the probability measure

$$\frac{2}{\pi}\sqrt{1-t^2}\,dt.$$



Cohomology of Shimura varieties

A use of the FL from Morel's book. The L-function of the intersection complex IC^V of the Baily-Borel compactification of the Shimura variety attached to a unitary group takes the general form:

$$\log L_{\mathcal{P}}(s, IC^K V) =$$
$$\sum_H \sum_{\pi_H} \sum_{r_H} c_H(\pi_H, r_H) \quad \log L(s - d/2, \pi_{H,\mathcal{P}}, r_H).$$

Ranks of elliptic curves

A use of the FL in Bhargava and Shankar (2010)

Let E be an elliptic curve over $\mathbb{Q}$. Let r be the rank of $E(\mathbb{Q})$. Let t be its analytic rank: $L(E,s) = (s-1)^t + \cdots$. Then $r = t$ for a positive fraction of all elliptic curves (ordered by height). (partial Birch-Swinnerton-Dyer)

Recent uses of the FL.

- Morel and Shin (Shimura varieties) $\rightarrow$ Skinner and Urban (Iwasawa conjectures) $\rightarrow$ Bhargava and Shankar (BSD: for a positive fraction of elliptic curves)
- Shimura varieties $\rightarrow$ Sato-Tate (Clozel-Harris-Shepherd-Barron-Taylor) $\rightarrow$ better Sato-Tate (Barnet-Lamb -Geraghty - Harris - Taylor)
- Classification of automorphic representations of classical groups (Arthur)