(9)

MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im **Email/Phone:** mim2@illinois.edu

Speaker's Name: Victor Ginzburg

Talk Title: Geometry of Quiver Varieties

Date: 9/3/14 **Time:** 2:00 am / pm (circle one)

List 6-12 key words for the talk: Geometric invariant theory for quiver varieties, local systems on quivers

Please summarize the lecture in 5 or fewer sentences: Ginzburg reminds the audience of GIT for quivers, the moment map in the case for the 1-Jordan quiver, and a connection between g-loop quiver with a smooth Riemann surface C of genus g. He talks about the Hilbert-Mumford criterion and proves that the proj morphism $\mathcal{M}_\lambda(V,W) \to \mathcal{M}_\lambda(V,W) = \mathcal{M}_0(V,W)$ is semi-small.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>Computer Presentations</u>: Obtain a copy of their presentation
  - <u>Overhead</u>: Obtain a copy or use the originals and scan them
  - <u>Blackboard</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - <u>Handouts</u>: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to <u>notes@msri.org</u> with the workshop name and your name in the subject line.

(9)

Geometry of Quiver Varieties
Victor Ginzburg
Wed, Sept 3, 2014, 2-3 pm

GIT Reminder

Stability: G reductive group acting on X affine.
$\chi : G \to \mathbb{C}^*$ char.

$$\mathbb{C}[X]^\chi = \chi\text{-semi-invariants}$$

$$A_\chi = \bigoplus_{\substack{n \geq 0 \\ n \in \mathbb{Z}}} \mathbb{C}[X]^{\chi^n}$$

$$X/\!/_\chi G := \mathrm{Proj}(A_\chi)$$

Special case where $\chi = 1$,

$$X/\!/_1 G = X/\!/G = \mathrm{Spec}(\mathbb{C}[X]^G) \text{ affine}$$

$$\pi : X/\!/_\chi G \xrightarrow[\text{morphism}]{\text{proj}} X/\!/G$$

I - finite set
V_i - f.d. v. space
$V = (V_i)_{i \in I}$

$$G_V = \prod_{i \in I} GL(V_i), \quad \mathfrak{g}_V = \bigoplus_{i \in I} \mathfrak{gl}(V_i)$$

chars. of G_V $\cong \mathbb{Z}^I$

$$\mathrm{Hom}(G_V, \mathbb{C}^*) = \left\{ \begin{array}{l} \chi : \prod_{i \in I} GL(V_i) \to \mathbb{C}^*, \\ (g_i) \longmapsto \prod_{i \in I} \det(g_i)^{\chi_i} \end{array} \right\}$$

$$\cong \mathbb{Z}^I = \{(\chi_i)_{i \in I}\}, \quad \chi_i \in \mathbb{Z}$$

$$(\mathfrak{g}_V^*)^{G_V} = \left\{ \begin{array}{l} \lambda : \bigoplus_{i\in I} \mathfrak{gl}(V_i) \xrightarrow{\text{linear}} \mathbb{C}, \text{ conjugation inv,} \\ \qquad (u_i) \longmapsto \sum_{i\in I} \lambda_i \operatorname{Tr}(u_i) \end{array} \right\}$$

$$\cong \mathbb{C}^I = \{ \underbrace{(\lambda_i)_{i\in I}}_{\lambda} \}$$

Q a quiver with vertex set I,
we define Q^{op}:
edge $i \xrightarrow{x\in Q} j \longleftrightarrow j \xrightarrow{x^*} i$

$\overline{Q} = Q \cup Q^{op}$ (called the double of Q)

Observe $\displaystyle \operatorname{Rep}(Q,V) = \prod_{\substack{i \xrightarrow{e} j \\ \text{edges}}} \operatorname{Hom}(V_i, V_j)$

$$\operatorname{Rep}(Q,V)^* = \operatorname{Rep}(Q^{op}, V)$$

$$\begin{aligned} T^* \operatorname{Rep}(Q,V) &= \operatorname{Rep}(Q,V) \times \operatorname{Rep}(Q,V)^* \\ &= \operatorname{Rep}(Q,V) \times \operatorname{Rep}(Q^{op},V) \\ &= \operatorname{Rep}(\overline{Q}, V) \end{aligned}$$

Moment map
$$\begin{array}{ccc} \mu : T^* \operatorname{Rep}(Q,V) & \longrightarrow & \mathfrak{g}_V^* \\ \| \text{ tr} & & \wr\| \text{ tr} \\ \operatorname{Rep}(\overline{Q}, V) & & \mathfrak{g}_V = \bigoplus_{i\in I} \mathfrak{gl}(V_i) \end{array}$$

<u>Lemma</u> $\mu : \operatorname{Rep}(\overline{Q}, V) \longrightarrow \mathfrak{g}_V$ has the form
$$\begin{array}{c} (\underset{\substack{\cap \\ Q}}{x}, \underset{\substack{\cap \\ Q^{op}}}{x^*}) \longmapsto \displaystyle\sum_{x\in Q} x x^* - x^* x \end{array}$$

For $i \in I$

$$\left(\sum [x, x^*] \right)_i = \sum x_{\text{in } i}\, x^*_{\text{out } i} - \sum x^*_{\text{in } i}\, x_{\text{out } i}.$$

<u>Proof</u> in the case of Jordan quiver

$Q \qquad \bullet \circlearrowleft x$

$\overline{Q} \qquad \bullet \circlearrowleft x, \ \circlearrowleft x^*$

$$\mathrm{Rep}(\overline{Q}, V) = \mathfrak{gl}_V \times \mathfrak{gl}_V$$

$$\mu : \mathfrak{gl}_V \times \mathfrak{gl}_V \longrightarrow \mathfrak{gl}_V^* \cong \mathfrak{gl}_V$$

$$\begin{aligned} \mu(x, x^*) &= [u \mapsto \langle x^*, \operatorname{ad} u(x) \rangle] \\ &= [u \mapsto \operatorname{tr}(x^*[u, x])] \\ &= [u \mapsto \operatorname{tr}(u[x^*, x])] \end{aligned}$$

So $\mu : (x, x^*) \longmapsto [x, x^*]$.

The argument for a general quiver is the same.

<u>Defn</u> Given a quiver Q, v. space V, $\chi = (\chi_i) \in \mathbb{Z}^I$, $\lambda = (\lambda_i) \in \mathbb{C}^I$,

$$\mathcal{M}_{\chi, \lambda}(Q, V) := \mu^{-1}(\lambda) /\!/_{\chi} G_V \quad \text{(quiver variety)}.$$

<u>Examples</u> $\mathcal{M}_{\chi,\lambda}(G^\bullet, V) = \begin{cases} \phi & \text{if } \lambda \neq 0 \text{ or } \chi \neq 0, \\ \text{Commuting var} /\!\!/ GL_V = \dfrac{\mathbb{C}^n \times \mathbb{C}^n}{S_n} \end{cases}$

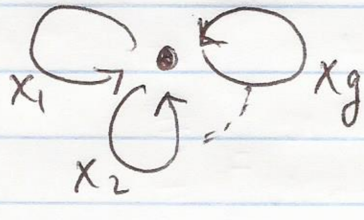
can simultaneously diagonalize 2 commuting matrices

$$\mu : \mathfrak{gl}_V \times \mathfrak{gl}_V \longrightarrow \mathfrak{gl}_V \ni \lambda = \lambda \cdot Id$$
$$(x, x^*) \longmapsto [x, x^*]$$

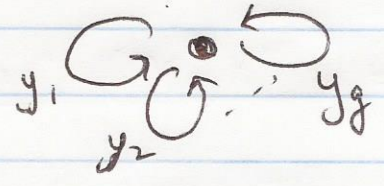
$\lambda = (\lambda_i)_{i \in I} = \lambda$ since $I = \{pt\}$

$$\mu^{-1}(\lambda) = \{(x, x^*) : [x, x^*] = \lambda \cdot Id\}$$

$\mu^{-1}(0)$ = commuting variety
= the space of pairs of matrices that commute.

Q 

g loops

Q^{op} 

g loops

$\overline{Q}$ $2g$ loops at one vertex
$x_1, \ldots, x_g, y_1, \ldots, y_g$

$$\mu : (x_1, \ldots, x_g, y_1, \ldots, y_g) \longmapsto \sum_{i=1}^{g} [x_i, y_i]$$

$$\mu^{-1}(0) = \{(x_1, \ldots, x_g, y_1, \ldots, y_g) : \sum_{i=1}^{g} [x_i, y_i] = 0\}$$

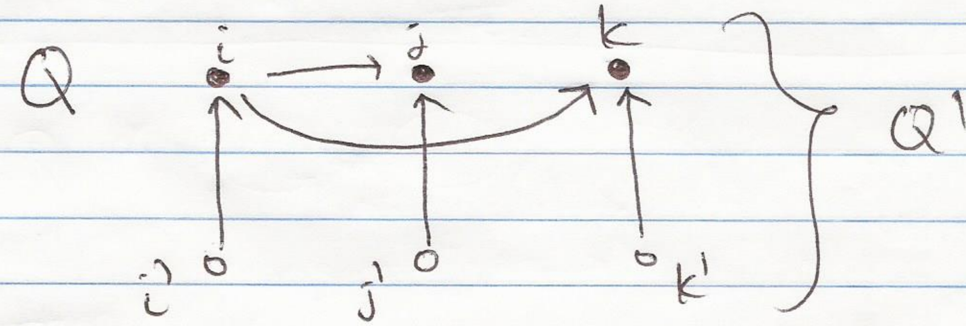
since $\mu^{-1}(\lambda) = \phi \quad \forall \lambda \neq 0$.

Let C be a smooth Riemann surface of genus g.

$$\pi_1(C) = \{ a_1, \ldots, a_g, b_1, \ldots, b_g : \prod_{\alpha=1}^{g} [a_\alpha, b_\alpha] = 1 \}.$$

$$\begin{array}{c} T_{triv}\Big(\underset{\text{(up to isom)}}{\mathrm{Rep}_n\, \pi_1(C)}\Big) = T_{triv}\Big(\text{rank } n \text{ local systems on } C \text{ (up to isom)}\Big) \\ \| \\ \mu^{-1}(0)/GL_n =: \underset{\text{affine quotient}}{\mathcal{M}_0(Q, \mathbb{C}^n)}. \end{array}$$

Given a quiver Q, define Q' (framed quiver)



$\mathrm{Rep}(\bar{Q}', V, W)$

$G_V \curvearrowright \mathrm{Rep}(\bar{Q}', V, W)$, so does G_W, but forget about G_W-action.

Have

$$T^* \mathrm{Rep}(Q', V, W) \longrightarrow \mathfrak{g}_V^*$$

$$\mathcal{M}_{\chi, \lambda}(Q, V, W) = \mu^{-1}(\lambda) /\!/_{\chi}\, G_V$$

(Nakajima quiver variety)

An elmt of $\mathrm{Rep}(\overline{Q}', V, W)$ is given by (x, x^*, i, i')

$$\begin{tikzcd} \bullet \arrow[d, bend left, "i'"] \\ \bullet \arrow[u, bend left, "i"] \end{tikzcd}$$

$$(x, x^*, i, i') \xmapsto{\mu} \sum [x, x^*] + i i'$$

$\mu^{-1}(\lambda)^s := \{ (x, x^*, i, i') \in \mu^{-1}(\lambda) :$ for any $V' \subseteq V$ s.t. $V' \supseteq \mathrm{im}(i)$ and V' is (x, x^*)-stable, then $V' = V \}$.

<u>Hilbert-Mumford</u>

$\Rightarrow$ <u>Thm</u> 1.) $\mu^{-1}(\lambda)^s$ is smooth $\forall \lambda$ ~~2.~~ and $G_V \circlearrowright$ freely on $\mu^{-1}(\lambda)^s$

2.) $\mathcal{M}_\lambda(V, W) \cong \mu^{-1}(\lambda)^s / G_V$ sm, not affine in general.

From now on, restrict $\chi_i = 1 \ \forall i$, i.e., $\chi : (g_i) \longmapsto \prod_{i \in I} \det(g_i)$.

Can also consider $\mu^{-1}(\lambda) /\!\!/ G_V$ (categorical quotient)

$$\mu^{-1}(\lambda) /\!\!/ G_V = \overline{\mathcal{M}}_\lambda(V, W)$$

affine, but not always smooth.

$\chi \equiv 1, \lambda$

$$\pi : \mathcal{M}_\lambda(V, W) \xrightarrow{\text{proj.}} \overline{\mathcal{M}}_\lambda(V, W)$$

<u>Rmk</u> For λ general enough, π is an isomorphism.

Thm. π is semi-small, i.e.,

$$\dim(\mathcal{M}_\lambda \times_{\overline{\mathcal{M}}_\lambda} \mathcal{M}_\lambda) \le \dim \mathcal{M}_\lambda .$$

Proof

Reformulation

π is semi-small $\Leftrightarrow$ $\pi_* \mathbb{C}_{\mathcal{M}_\lambda}[\dim \mathcal{M}_\lambda]$ is a perverse sheaf.

$\lambda \in \mathbb{C}^I$.

Take a generic line ℓ through 0 in $\mathbb{C}^I$.

$$\begin{tikzcd}
\mathcal{M}_0 \arrow[r, hook] \arrow[d, "\pi_0"'] & \mu^{-1}(\ell)/\!\!/_{\chi} G_V = \mathcal{M} \arrow[d] \\
\overline{\mathcal{M}}_0 \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}} \arrow[d] \\
\{0\} \arrow[r, hook] & \ell
\end{tikzcd}$$

☆ Show that the push-forward of a constant sheaf is perverse.

same constructions as $\mathcal{M}_\lambda$ and $\overline{\mathcal{M}}_\lambda$ but use $\mu^{-1}(\ell)$, where ℓ is a line, instead of $\mu^{-1}(\lambda)$, where λ is a point.

η generic point.

$$\begin{tikzcd}
\mathcal{M}_0 \arrow[r, hook, "\text{sm family}"'] \arrow[d, "\pi_0"'] & \mathcal{M} \arrow[d, "\pi"] & \mathcal{M}^\circ \arrow[l, hook'] \arrow[d, "\pi_\eta"] \\
\overline{\mathcal{M}}_0 \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}} \arrow[d, "f"] & \overline{\mathcal{M}}^\circ \arrow[l, hook'] \arrow[d, "f_\eta"] \\
\{0\} \arrow[r, hook] & \Delta & \mathbb{C}^* \arrow[l, hook']
\end{tikzcd}$$

Use the fact that the functor of nearby cycles takes perverse sheaves to perverse sheaves.

$$(\pi_0)_* \mathbb{C}_{\mathcal{M}_0} = \pi_* \psi_{f\circ\pi} \mathbb{C}_{\mathcal{M}^\circ}$$

$$= \psi_f \underbrace{(\pi_\eta)_* \mathbb{C}_{\mathcal{M}^\circ}}_{\text{const. sh. on sm. mfd.}}$$

$$= \psi_f (\mathbb{C}_{\overline{\mathcal{M}}_0})$$

is perverse (up to a number of homological shifts).

$\boxtimes$