

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im Email/Phone: mim2@illinois.edu

Speaker's Name: Pramod Achar

Talk Title: The Springer Correspondence

Date: 9, 4, 14 Time: 10:30 (am) / pm (circle one)

List 6-12 key words for the talk: Springer sheaf, hyperbolic localization, Mackey formula, cuspidal pair

Please summarize the lecture in 5 or fewer sentences: Achar defines induced and restricted functors for sheaves on the Levi to sheaves on G and for sheaves on G to sheaves on the Levi via $\text{Ind}_G^L = R\Gamma^* G \times^L G/P$ and $\text{res}_G^L = R\Gamma^! \mathbb{P}$, where the pullback and the push-forward maps are given on the first page of the lecture notes. Achar discusses the Springer correspondence for the group setting, with a thorough discussion on certain missing parts in the correspondence between $\text{Irr}(W)$ and certain simple summands in $\text{Irr}(\text{Per}_G(W))$.

CHECK LIST

(This is NOT optional, we will not pay for incomplete forms)

- Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
 - **Computer Presentations:** Obtain a copy of their presentation
 - **Overhead:** Obtain a copy or use the originals and scan them
 - **Blackboard:** Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
 - **Handouts:** Obtain copies of and scan all handouts
- For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (YYYY.MM.DD.TIME.SpeakerLastName)
- Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

The Springer Correspondence

Pramod Achar

Thurs, Sept 4, 2014, 10:30-11:30 am

$$\text{Let } X_G = \{ (C, \epsilon) \} = \text{Irr}(\text{Per}_{U_G}(U))$$

\uparrow unip. class \uparrow irredu
 G -equiv. loc. sys.

$P = LV$, P parabolic subgroup

$$\text{ip}: P \xrightarrow{\text{incl}} G, \quad \text{qp}: P \xrightarrow{\text{projection}} L$$

$$\text{res}_{L \subseteq P}^G = R\text{qp}! \text{ip}^*$$

$$\text{ind}_{L \subseteq P}^G = R\pi_* G \times^P \text{qp}!$$

$$G \times^P P \xrightarrow{\pi} G$$

Say $(C_1, \epsilon_1) \in X_L$.

$\text{IC}(C_1, \epsilon_1)$ supp on $\bar{C}_1 \subseteq U_L \subseteq L$

$$\text{qp}^! (\quad) \quad \ll \quad \bar{C}_1 \vee \subseteq P$$

$$G \times^P \bar{C}_1 \vee \xrightarrow{\downarrow \pi} U_C \quad \text{qp}^! \text{IC}(C_1, \epsilon_1)$$

For $(C_i, \epsilon_i) \in X_L$,

$$\text{ind}_{L \in P}^G IC(C_i, \epsilon_i)$$

is supp on U_G .

Yesterday

$$\text{ind}_{T \in B}^G IC(C_e, \underline{\mathbb{1}}) = A = \bigoplus_{(C, \epsilon) \in X_G} IC(C, \epsilon) \otimes V_{C, \epsilon}$$

Springer sheaf

- either 0 or an
irredu. W rep

- Every irredu
 W rep occurs
once.

Get

$$\text{Irr}(W) \xleftrightarrow{\sim} \left\{ \begin{array}{l} \text{simple} \\ \text{summands} \\ \text{of } A \end{array} \right\} \subseteq X_G$$

Problem: What about the missing parts?

Ex. $G = SL_4$

$$|X_G| = 9, \quad |\text{Irr}(S_4)| = 5$$

4 missing pairs.

Idea: Develop a "Harish-Chandra theory" for $\text{Per}_{V_G}(U)$.

Thm (Q)

- (res, ind) is an adjoint pair.
- They restrict to exact functors.

$$\text{Per}_{V_L}(U_L) \rightleftarrows \text{Per}_{V_G}(U_G)$$

- They take s.s. perov. sh. to s.s. perov. sh.

Proof (sketch)

Exactness

- Can get a 1-sided exactness stmt by dim cal. like in Thm 4(b).
- Braden's theory of hyperbolic localization.

Semisimplicity

- ind : decomposition thm
- res : Braden.

Side note: $\text{Per}_{V_G}(U_G)$ is semi simple!

Defn $(C, \varepsilon) \in X_G$ is cuspidal if
 $\text{res}_{L \leq P}^G \text{IC}(C, \varepsilon) = 0 \quad \forall P \subsetneq G.$

Thm (7), ("Induction series")
 $\forall (C, \varepsilon) \in X_G, \exists (L, C_0, \varepsilon_0)$ where
 L Levi, $(C_0, \varepsilon_0) \in X_L$ cuspidal

s.t. $\text{IC}(C, \varepsilon)$ is a summand of $\text{ind}_{L \leq P}^G \text{IC}(C_0, \varepsilon_0)$.
 unique up to G -conjugacy

Pf (sketch)

Existence: obvious

Uniqueness: use "Mackey formula", that relates $\text{res} \circ \text{ind}$ to $\text{ind} \circ \text{res}$.

So

$$(*) \quad X_G = \coprod_{(L, C_0, \varepsilon_0)} X_G^{(L, C_0, \varepsilon_0)}$$

$$X_G^{(L, C_0, \varepsilon_0)} = \text{simple summands of } \text{ind}_{L \leq P}^G \text{IC}(C_0, \varepsilon_0)$$

One piece of (*):

$$X_G^{(T, \{e\}, \emptyset)} = \text{simple summands of } A \leftrightarrow \text{Irr}(W)$$

Thm. (8) $\forall (L, C_0, \epsilon_0),$

$$X_G^{(L, C_0, \epsilon_0)} \longleftrightarrow \text{Irr}(N_G(L)/L)$$

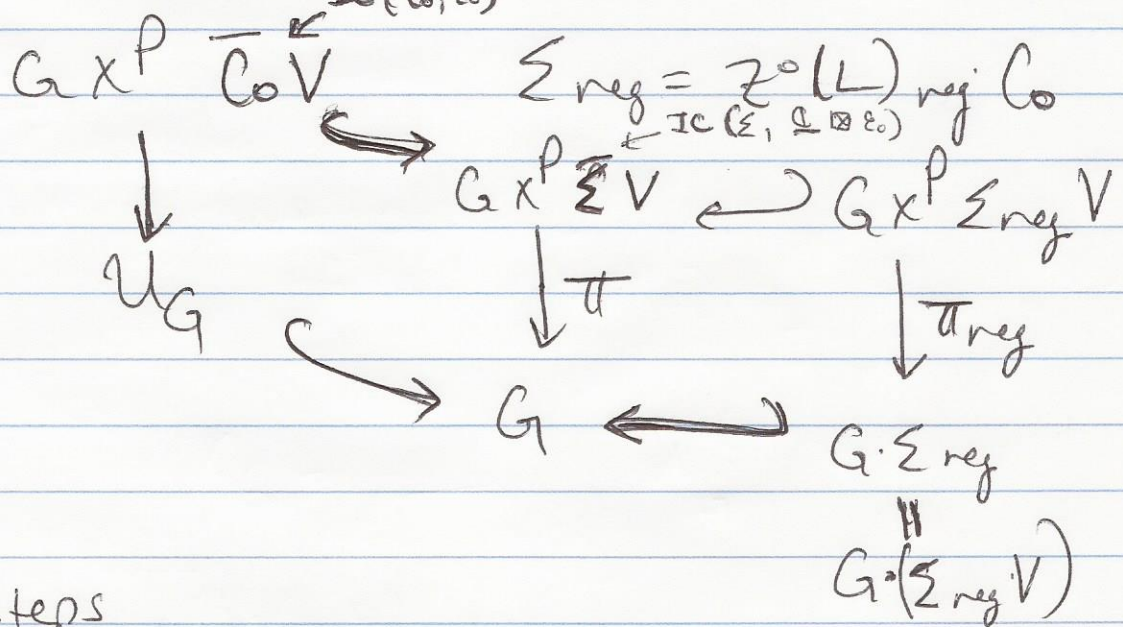
"relative Weyl group"

Pf (sketch) $P = LV$

$$Z^\circ(L) = \text{conn. comp. of center of } L.$$

$$Z^\circ(L)_{\text{reg}} = \{ x \in Z^\circ(L) : C_G^\circ(x) \subseteq L \}$$

Main diagram: $\Sigma = Z^\circ(L) C_0$



steps

1.) $G \times P \Sigma_{\text{reg}} V \cong \{ (g, xL) \in G \times G/L : x^{-1}gx \in \Sigma_{\text{reg}} \}$

$N_G(L)/L$ acts on this.

2.) π_{reg} is a Galois covering map
 w/ $G_P = \text{NG}(L)/L$.

$$\text{Let } \mathcal{L} = (\pi_{\text{reg}})_* (G \times^P G_P^* (\mathbb{C} \otimes \varepsilon_0))$$

3.) $\text{End}(\mathcal{L}) = \mathbb{C}[\text{NG}(L)/L]$

most delicate... need to understand what (C_0, ε_0) can be.

4.) $\mathcal{L} \cong \pi_{\text{reg}}_* (G \times^P (j_! \text{IC}(C, \varepsilon, \mathbb{C} \otimes \varepsilon_0)))$

$$\cong j_! * \mathcal{L}[\dots]$$

restrict to U_G
 $\text{ind}_{L \subset P}^G \text{IC}(C_0, \varepsilon_0)$

5.) The map
 $\mathbb{C}[\text{NG}(L)/L] = \text{End}(j_! * \mathcal{L}) \rightarrow \text{End}(\text{ind IC}(C_0, \varepsilon_0))$

is an isom.

$$\begin{aligned} 6.) \text{ind IC}^L(C_0, \varepsilon_0) &= \bigoplus_{(C, \varepsilon)} \text{IC}^G(C, \varepsilon) \otimes V_{C, \varepsilon} \\ &= \bigoplus_{V \in \text{Irr}(\text{NG}(L)/L)} (\text{simple obj}) \otimes V \end{aligned}$$

Match terms.



Combine Thms ⑦ + ⑧

$$\coprod_{(L, \mathcal{C}, \mathcal{E}, \mathcal{E}_0)} \text{Irr}(N_G(L)/L) \cong X_G = \{(C, \mathcal{E})\}$$

Generalized Springer correspondence

$$SL_4 : \begin{cases} |X_G^{(T, 1, \mathcal{C})}| = 5 & |X_G| = 9 \\ |X_G^{(A_1 \times A_1, \dots)}| = 2 \end{cases}$$

+ 2 cuspidal pairs.

Context: character sheaves (~ 1980's)

- Certain G -equiv. perv. sh. on G .

- Goal for G / \mathbb{F}_q

→ compute character values of finite Chevalley gps $G(\mathbb{F}_q)$.

- Perv $G(U_G)$ - usually NOT char. sh. ($\mathcal{F} \neq \mathcal{H} \stackrel{\text{is}}{=} \text{a char sheaf}$)

Lübeck

More or less:

Reduce to compute stalks of IC's on U_G .

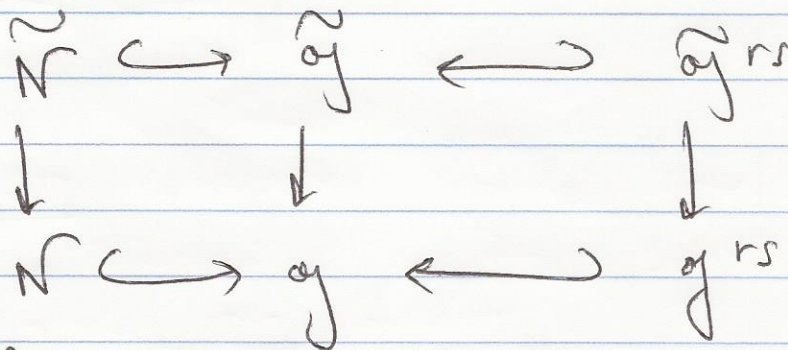
Stalks of IC's on U_G :
 "Lusztig-Shoji algorithm."

- Explicit knowledge of gen. Springer
- Basic rep. th. of $N_{\mathbb{A}}(L)/L$.
- How to manipulate matrices.

Lie algebra version

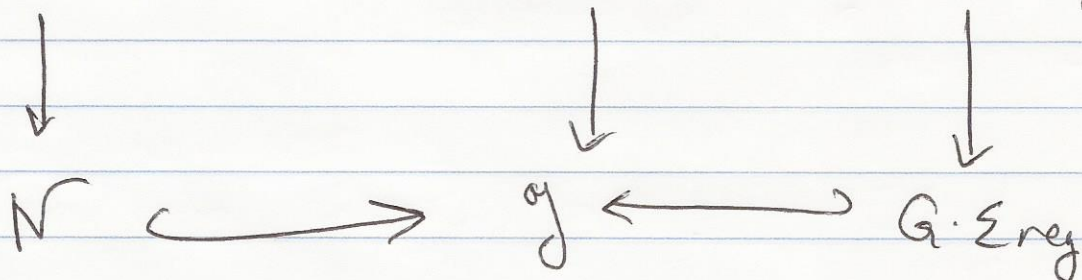
N = nilpotent cone $\subseteq \mathfrak{g}$

Wed's main diagram:



$$\mathfrak{g} = \mathfrak{l} + \mathfrak{p}, \quad \Sigma = \mathfrak{C}_0 + \mathfrak{Z}_L$$

$$G \times P(\bar{\mathfrak{C}}_0 + \mathfrak{D}) \hookrightarrow G \times P(\bar{\Sigma} + \mathfrak{D}) \twoheadrightarrow G \times P(\Sigma_{reg} + \mathfrak{D})$$



Transfer info:

$$(j_! * L) |_{\mathbb{R}} \cong \text{ind IC}(C_0, E_0)$$

OR Fourier transform
Deligne
Sato

$$T : \text{Per}_{U_G}(g) \rightarrow \text{Per}_G(g)$$

Prop $T(j_! * L) = \text{ind IC}(C_0, E_0)$

$$\mathbb{Q}[Na(L)] = \text{End}(j_! * L)$$

