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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im **Email/Phone:** mim2@illinois.edu

Speaker's Name: Pramod Achar

Talk Title: The Springer Correspondence

Date: 9/4/14 **Time:** 10:30 (am)/ pm (circle one)

List 6-12 key words for the talk: Springer sheaf, hyperbolic localization, Mackey formula, cuspidal pair

Please summarize the lecture in 5 or fewer sentences: Achar defines induced and restricted functors for sheaves on the Levi to sheaves on G and for sheaves on G to sheaves on the Levi via $\text{ind}_{L\subseteq P}^{G} = R\pi_* \, G\times^P g_!$ and $\text{res}_{L\subseteq P}^{G} = Rg_{P!}\, i_P^*$, where the pullback and the push-forward maps are given on the first page of the lecture notes. Achar discusses the Springer correspondence for the group setting, with a thorough discussion on certain missing parts in the correspondence between $\text{Irr}(W)$ and certain simple summands in $\text{Irr}(\text{Perv}_G(\mathcal{U}))$.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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The Springer Correspondence
Pramod Achar
Thurs, Sept 4, 2014, 10:30 – 11:30 am

Let $X_G = \{(C, \mathcal{E})\} = \operatorname{Irr}(\operatorname{Perv}_G(\mathcal{U}))$

C ← unip. class
$\mathcal{E}$ ← irredu G-equiv. loc. sys.

$P = LV$, P parabolic subgroup

$i_P : P \xrightarrow{\text{incl}} G$, $q_P : P \xrightarrow{\text{projection}} L$

$\operatorname{res}^G_{L \subseteq P} = R q_{P!} i_P^*$,

$\operatorname{ind}^G_{L \subseteq P} = R\pi_* \, G \times^P q_P^!$

$G \times^P P \xrightarrow{\pi} G$

Say $(C_1, \mathcal{E}_1) \in X_L$.

$IC(C_1, \mathcal{E}_1)$ supp on $\overline{C_1} \subseteq \mathcal{U}_L \subseteq L$

$q_P^!(\quad " \quad)$ " " $\overline{C_1} V \subseteq P$

$G \times^P \overline{C_1 V}$ ← $q_P^! IC(C_1, \mathcal{E}_1)$
$\downarrow \pi$
$\mathcal{U}_G$

For $(C_1, \mathcal{E}_1) \in X_L$,

$\mathrm{ind}_{L \subseteq P}^{G} IC(C_1, \mathcal{E}_1)$

is supp on $\mathcal{U}_G$.

Yesterday

$$\mathrm{ind}_{T \subseteq B}^{G} IC(C_e, \underline{\mathbb{1}}) = A = \bigoplus_{(C,\mathcal{E}) \in X_G} IC(C,\mathcal{E}) \otimes V_{C,\mathcal{E}}$$

Springer sheaf ↗ (A)

- either 0 or an irredu. W rep
- Every irredu W rep occurs once.

Get

$$\mathrm{Irr}(W) \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{simple} \\ \text{summands} \\ \text{of } A \end{array} \right\} \subseteq X_G$$

<u>Problem</u>: What about the missing parts?

<u>Ex</u>. $G = SL_4$

$$|X_G| = 9, \quad |\mathrm{Irr}(S_4)| = 5$$

4 missing pairs.

Idea: Develop a "Harish-Chandra theory" for $\mathrm{Perv}_G(\mathcal{U})$.

Thm (6).
- (res, ind) is an adjoint pair.
- They restrict to exact functors.

$$\mathrm{Perv}_L(\mathcal{U}_L) \rightleftarrows \mathrm{Perv}_G(\mathcal{U}_G)$$

- They take s.s. perv. sh to s.s. perv. sh.

Proof (sketch)

Exactness
- Can get a 1-sided exactness stmt by dim cal. like in Thm 4(b).
- Braden's theory of hyperbolic localization.

Semisimplicity
- ind: decomposition thm
- res: Braden.

Side note: $\mathrm{Perv}_G(\mathcal{U}_G)$ is semisimple!

Defn $(C, \mathcal{E}) \in X_G$ is *cuspidal* if

$$\mathrm{res}^G_{L \subseteq P} \, IC(C, \mathcal{E}) = 0 \quad \forall P \subsetneq G.$$

Thm (7). ("Induction series")
$\forall \, (C, \mathcal{E}) \in X_G, \ \exists \, (L, C_0, \mathcal{E}_0)$ where
L Levi, $(C_0, \mathcal{E}_0) \in X_L$ cuspidal

s.t. — unique up to G-conjugacy
$IC(C, \mathcal{E})$ is a summand
of $\mathrm{ind}^G_{L \subseteq P} \, IC(C_0, \mathcal{E}_0)$.

Pf (sketch)
Existence: obvious
Uniqueness: use "Mackey formula", that relates $\mathrm{res} \circ \mathrm{ind}$ to $\mathrm{ind} \circ \mathrm{res}$. ...

So

$$(*) \qquad X_G = \coprod_{(L, C_0, \mathcal{E}_0)} X_G^{(L, C_0, \mathcal{E}_0)}$$

$X_G^{(L, C_0, \mathcal{E}_0)}$ = simple summands of $\mathrm{ind}^G_{L \subseteq P} \, IC(C_0, \mathcal{E}_0)$.

One piece of $(*)$:
$X_G^{(T, \{e\}, \underline{\mathbb{C}})}$ = simple summands of A $\longleftrightarrow$ $\mathrm{Irr}(W)$

Thm. (8) $\forall (L, C_0, \mathcal{E}_0)$,

$$X_G^{(L, C_0, \mathcal{E}_0)} \longleftrightarrow \mathrm{Irr}(\underbrace{N_G(L)/L}_{\text{"relative Weyl group"}})$$

Pf (sketch) $P = LV$

$Z^\circ(L)$ = conn. comp. of center of L.

$Z^\circ(L)_{reg} = \{ x \in Z^\circ(L) : C^\circ_G(x) \subseteq L \}$.

Main diagram: $\Sigma = Z^\circ(L) C_0$

$\Sigma_{reg} = Z^\circ(L)_{reg} C_0$

$$\begin{array}{ccccc}
G \times^P \overline{C_0 V} \; \scriptstyle{(IC(C_0, \mathcal{E}_0))} & \hookrightarrow & G \times^P \overline{\Sigma} V \; \scriptstyle{(IC(\Sigma, \mathbb{1} \boxtimes \mathcal{E}_0))} & \hookleftarrow & G \times^P \Sigma_{reg} V \\
\downarrow & & \downarrow \pi & & \downarrow \pi_{reg} \\
\mathcal{U}_G & \hookrightarrow & G & \hookleftarrow & G \cdot \Sigma_{reg} \\
 & & & & \parallel \\
 & & & & G \cdot (\Sigma_{reg} V)
\end{array}$$

Steps

1.) $G \times^P \Sigma_{reg} V \cong \{ (g, xL) \in G \times G/L : x^{-1} g x \in \Sigma_{reg} \}$

$N_G(L)/L$ acts on this.

2.) π_{reg} is a Galois covering map w/ gp $= N_G(L)/L$.

Let $\mathcal{L} = (\pi_{reg})_*(G \times^P q_P^*(\mathbb{C} \boxtimes \mathcal{E}_0))$

3.) $End(\mathcal{L}) = \mathbb{C}[N_G(L)/L]$

most delicate.... need to understand what $(C_0, \mathcal{E}_0)$ can be,

4.) $R\pi_* \, G \times^P (q_P^! \, IC(\Sigma, \mathbb{C} \boxtimes \mathcal{E}_0))$

$= j_{!*} \mathcal{L}[\text{---}]$.

restrict to $\mathcal{U}_G$:

$ind_{L \subseteq P}^G IC(C_0, \mathcal{E}_0)$

5.) The map

$\mathbb{C}[N_G(L)/L] = End(j_{!*}\mathcal{L}) \to End(ind \, IC(C_0, \mathcal{E}_0))$

is an isom.

6.) $ind \, IC^L(C_0, \mathcal{E}_0) = \bigoplus_{(C, \mathcal{E})} IC^G(C, \mathcal{E}) \otimes V_{C, \mathcal{E}}$

$= \bigoplus_{V \in Irr(N_G(L)/L)} (\text{simple obj.}) \otimes V$

Match terms.

$\boxtimes$

Combine Thms (7) + (8)

$$\coprod_{(L, C_0, \mathcal{E}_0)} \mathrm{Irr}\left(N_G(L)/L \right) \xrightarrow{\sim} X_G = \{(C, \mathcal{E})\}$$

Generalized Springer correspondence

SL_4 : $\left| X_G^{(T, 1, \mathbb{C})} \right| = 5$ $\quad |X_G| = 9$

$\left| X_G^{(A_1 \times A_1, \ldots)} \right| = 2$

+ 2 cuspidal pairs.

Context: character sheaves (~ 1980's)

- Certain G-equiv. perv. sh. on G.
- Goal for $G/\mathbb{F}_q$
  - ⇝ compute character values of finite Chevalley gps $G(\mathbb{F}_q)$.
- $\mathrm{Perv}_G(\mathcal{U}_G)$ – usually NOT char. sh. ($j_{!*}\mathcal{L}$ is a char sheaf)

Lübeck

More or less:
Reduce to compute stalks of IC's on $\mathcal{U}_G$.

Stalks of IC's on $\mathcal{U}_G$:
"Lusztig–Shoji algorithm."

- Explicit knowledge of gen. Springer
- Basic rep. thy of $N_G(L)/L$.
- How to manipulate matrices.

Lie algebra version

$\mathcal{N}$ = nilpotent cone $\subseteq \mathfrak{g}$

Wed's main diagram:

$$\begin{array}{ccccc} \tilde{\mathcal{N}} & \hookrightarrow & \tilde{\mathfrak{g}} & \hookleftarrow & \tilde{\mathfrak{g}}^{rs} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{N} & \hookrightarrow & \mathfrak{g} & \hookleftarrow & \mathfrak{g}^{rs} \end{array}$$

$\mathfrak{p} = \mathfrak{l} + \nu$, $\Sigma = C_0 + \mathfrak{z}_L$

$$\begin{array}{ccccc} G\times^P(\bar{C}_0 + \nu) & \hookrightarrow & G\times^P(\bar{\Sigma} + \nu) & \hookleftarrow & G\times^P(\Sigma_{reg} + \nu) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{N} & \hookrightarrow & \mathfrak{g} & \hookleftarrow & G\cdot\Sigma_{reg} \end{array}$$

Transfer info:

$$(j_{!*}\mathcal{L})|_{\mathcal{N}} \simeq \operatorname{ind} IC(C_0, \mathcal{E}_0)$$

OR Fourier transform

- Deligne
- Sato

$$T : Perv_G(\mathfrak{g}) \longrightarrow Perv_G(\mathfrak{g})$$

Prop $T(j_{!*}\mathcal{L}) = \operatorname{ind} IC(C_0, \mathcal{E}_0)$

$$\mathbb{Q}[N_G(L)/L] = End(j_{!*}\mathcal{L})$$

$(-)|_{\mathcal{N}}$: $\sim$ to $End(\operatorname{ind} IC(C_0, \mathcal{E}_0))$

T: $\sim$ to $End(IC(C_0, \mathcal{E}_0))$

these differ by the sign character