(14)

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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im **Email/Phone:** mim2@illinois.edu

Speaker's Name: Victor Ginzburg

Talk Title: Geometry of Quiver Varieties

Date: 9/4/14 **Time:** 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: Nakajima quiver variety, McKay Correspondence, tautological bundles on quiver varieties.

Please summarize the lecture in 5 or fewer sentences: Ginzburg reviews (Nakajima) quiver varieties and then discusses McKay correspondence for a finite subgroup of $SL_2(\mathbb{C})$. He gives the construction of tautological bundles on quiver varieties and then identifies that a quiver variety for affine Dynkin quiver is isom to torsion-free coherent sheaves on $(\mathbb{C}^2/\Gamma)_\lambda$, which is $M_\lambda(L,\mathbb{C})$, a Kleinian surface.

CHECK LIST

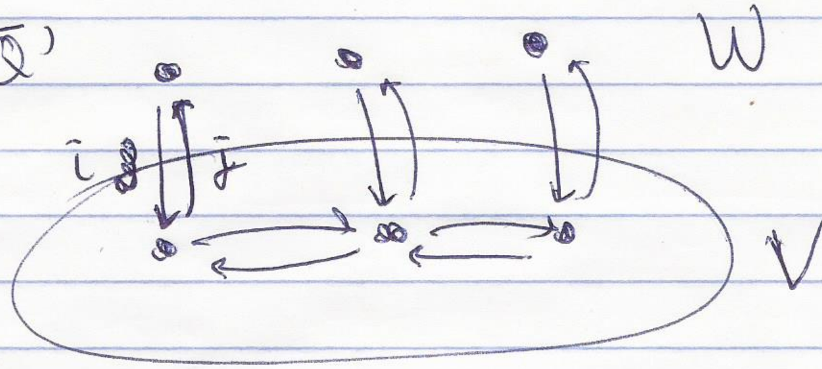
(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Geometry of Quiver Varieties
Victor Ginzburg
Thurs., Sept 4, 2014, 2-3pm

Q quiver, I vertex set
Q' framed quiver

$\overline{Q}'$



$\mathrm{Rep}(\overline{Q}', V, W)$
$\ni$
(x, x^*, i, j)

$\lambda = (\lambda_i)_{i \in I} \in \mathbb{C}^I$

$x : V_i \to V_j$
$x^* : V_j \to V_i$
$i : W_i \to V_i$
$j : V_i \to W_i$

$(\lambda_i \, Id_{V_i}) \in gl(V_i)$

$\lambda \in g_V = \bigoplus_i gl(V_i)$

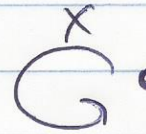
$\mathcal{M}_\lambda(V, W) = \mu^{-1}(\lambda) /\!\!/_{\chi} G_V,$

$G_V = \prod_{i \in I} GL(V_i)$

$\chi : G_V \longrightarrow \mathbb{C}^*$

$(g_i) \longmapsto \prod_{i \in I} \det(g_i)$

Ex 1.

 Q

 Q'

 $\overline{Q}'$

$\dim V = n$, $\dim W = 1$

$\overline{\mathcal{M}}_0 = \mathrm{Sym}^n\, \mathbb{C}^2$, $\mathcal{M}_0 = \mathrm{Hilb}^n(\mathbb{C}^2)$

$\mu(x, x^*, i, j) = [x, x^*] + ij$

$\mathcal{M}_0 \to \overline{\mathcal{M}}_0$ Hilb-Chow morphism

$$\begin{array}{l} \mathcal{M}_\lambda(V, W) \\ \quad \big\downarrow \pi \quad \text{proj. morphism} \\ \overline{\mathcal{M}}_\lambda(V, W) = \mu^{-1}(\lambda) /\!\!/ G_V \end{array}$$

affine.

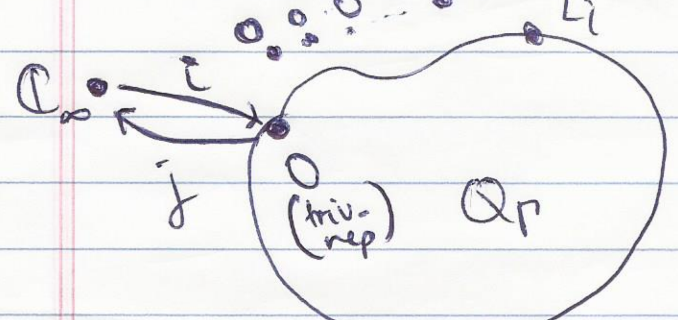
finite subgp

McKay correspondence $\Gamma \subseteq SL_2(\mathbb{C})$

$0 \longleftrightarrow$ triv.

$$\begin{matrix} I & = & \mathrm{Irrep}(\Gamma), \\ \Downarrow & & \Downarrow \\ i & \longleftrightarrow & L_i \end{matrix} \Bigg\} \rightsquigarrow e_{ij} = \#\{i \to j\} = \dim \mathrm{Hom}_\Gamma(L_i, L_j \otimes \mathbb{C}^2)$$

$(\mathrm{Irrep}\,\Gamma, \|e_{ij}\|) \rightsquigarrow Q_\Gamma$ McKay quiver ass. to Γ



$L = (L_i)$

(Q symmetric)

"Q is the double of its half".

Thm (Kronheimer)

1.) $\overline{\mathcal{M}_0(L, \mathbb{C})} = \mathbb{C}^2/\Gamma$

2.) $\overline{\mathcal{M}_\lambda(L, \mathbb{C})}$, $\lambda \in \mathbb{C}^{I \setminus \{0\}} \cong \mathfrak{h}$ Cartan subalg of the finite Dynkin diagram

Universal deformation of $\mathbb{C}^2/\Gamma$.

3.) $\mathcal{M}_\lambda(L, \mathbb{C}) \xrightarrow{\pi} \overline{\mathcal{M}_\lambda(L, \mathbb{C})}$

the minimal resolution of singularities

Ex. $\Gamma = \mathbb{Z}/(n)$, $\mathbb{C}^2/\Gamma = \{(x, y, z) : x^2 + y^2 + z^{n+1} = 0\}$

$$\mathbb{C}^2/\Gamma = \overline{\mathcal{M}_0(L, \mathbb{C})}$$

$$\overline{\mathcal{M}_\lambda(L,\mathbb{C})} = \{(x,y,z)\in\mathbb{C}^3 : x^2+y^2+z^{n+1}+\overset{\le n}{f(z)}=0\}$$

Tautological bundles on quiver varieties

$\mathrm{Rep}(\overline{Q}', L, M)$

$\forall\ i\in I$, have
$$\begin{array}{c} L_i \times \mathrm{Rep}(\overline{Q}', L, M) \\ \downarrow \\ \mathrm{Rep}(\overline{Q}', L, M) \end{array}$$

G_L-equiv. $\quad L_i \times \mu^{-1}(\lambda) \longrightarrow \mu^{-1}(\lambda)$

$$\mathcal{L}_i \xrightarrow{\mathrm{rk}(\dim L_i)} \mathcal{M}_\lambda(L,M)$$

nontriv. v. bdle. ass to $i\in Q_0$.

$i \xrightarrow{x} j \in \overline{Q}$, $\quad \mathcal{L}_i \xrightarrow{X} \mathcal{L}_j$ canonical morphism.

$\mathrm{Rep}(\overline{Q}', V, W)$

Monad: [Complex of length 3]
$$\bigoplus_{i\in I} \mathcal{L}_i\otimes V_i \xrightarrow{\sigma} \begin{pmatrix} \bigoplus_{i\to j} \mathcal{L}_i\otimes V_j \\ \oplus \\ \bigoplus_i \mathcal{L}_i\otimes W_i \end{pmatrix} \xrightarrow{\tau} \bigoplus_i \mathcal{L}_i\otimes V_i$$

$$\sigma : \begin{pmatrix} 1\otimes x - X\otimes 1 \\ 1\otimes j \end{pmatrix} \qquad \tau = \begin{pmatrix} 1\otimes x^* + X^*\otimes 1 \\ 1\otimes i \end{pmatrix}$$

This is a diagram of v. bdles on $\mathcal{M}_\lambda(V, W)$

(x, x^*, i, j)

$Q = \tilde{A}, \tilde{D}, \tilde{E}$, $L = \{L_i : \text{irreps of } \Gamma\}$
$M = \mathbb{C}$

<u>Lemma</u> (i) $\tau \circ \sigma = 0$
(ii) stability of (x, x^*, i, j)
$\Rightarrow$ τ surjective.
(iii) opp. stability
$\Rightarrow$ σ is injective.

<u>Thm</u> (Kronheimer & Nakajima)

Resoln of the deformation

$$\mathcal{M}_\lambda(V, W) \xrightarrow{\sim} \left\{ \begin{array}{l} \text{Torsion free coherent sheaves } \mathcal{F} \text{ on } \widetilde{(\mathbb{C}^2/\Gamma)}_\lambda = \mathcal{M}_\lambda(L, \mathbb{C}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{not compact} \\ \widetilde{(\mathbb{C}^2/\Gamma)}_\lambda \hookrightarrow \widetilde{(\mathbb{P}^2/\Gamma)} \\ \text{s.t. } \mathcal{F}|_{\infty/\Gamma} = W \otimes \mathcal{O} \end{array} \right\}$$

$$(x, x^*, i, j) \longmapsto \frac{\ker \tau}{\operatorname{im} \sigma}$$

From now on, $\underline{\underline{\lambda = 0}}$.

$$Q_\Gamma = \tilde{A}\tilde{D}\tilde{E}.$$

$$\widetilde{\mathbb{C}^2/\Gamma}$$

$\mathbb{P}^1, \ldots, \mathbb{P}^1$ with graph of ADE

$$\pi \downarrow$$

sm surface except at 0. $\mathbb{C}^2/\Gamma$

$$\mathbb{P}^1_i \cdot \mathbb{P}^1_i = -2.$$

$$I \setminus \{0\}, \quad Z_i \subseteq \mathcal{M}(V, W) \times \mathcal{M}(V \oplus \mathbb{C}_i, W)$$

$$\mathcal{F} \qquad \mathcal{F}'$$

$$Z_i = \{(\mathcal{F}, \mathcal{F}') : 0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{O}_{\mathbb{P}^1_i}(-1) \to 0\}$$

sm. Lagrangian subvariety

$\mathfrak{g}$ = Kac-Moody ass. to Q

ϖ_i fundamental weights.

a_i simple roots

$$\lambda = (\lambda_i) \in \mathbb{Z}^I \longleftrightarrow \lambda = \sum_i \lambda_i \varpi_i$$

$$\alpha = (\alpha_i) \in \mathbb{Z}^I \longleftrightarrow \alpha = \sum_i \alpha_i a_i$$

Thm 1) $\exists\ \mathcal{U}(\mathfrak{g}) \longrightarrow H^{BM}_{top}(\mathcal{M} \times_{\overline{\mathcal{M}}} \mathcal{M})$.

Z

2) $\Lambda_{V,W} = \pi^{-1}(0)$

$$\pi : \mathcal{M}(V,W) \longrightarrow \overline{\mathcal{M}}(V,W)$$

$\dim W = \lambda$

$\dim V = \alpha$

$H(\coprod_V \Lambda_{V,W}) = \text{Irrep of } \mathcal{U}(\mathfrak{g})$

is irrep of $\mathfrak{g}$, with highest weight λ.

$H(\Lambda_{W,V}) \longleftrightarrow$ weight space of weight $\lambda - \alpha$.

Irrep of $\mathfrak{g}$

Nakajima

IC sheaves of affine grassmanian.

symplectic duality

Geometric Satake.

(work-in-progress) Gaitto duality

[see Nick Proudfoot's talk 2014.09.05.1130.Proudfoot.pdf]

Zastava space

Φ function on $\mathrm{Rep}(\overset{\alpha}{V}, \overset{\lambda}{W}) \times \mathfrak{g}_V \times \mathfrak{g}_W$

$$\begin{tikzcd}
Z^\alpha \times \mathbb{A}^\lambda \arrow[d, "p_1"] & & \mathrm{Rep}(\overset{\alpha}{V}, \overset{\lambda}{W}) \times \mathfrak{g}_V \times \mathfrak{g}_W \arrow[dll, "p_2"'] \arrow[d] \\
\mathbb{A}^\alpha \times \mathbb{A}^\lambda & & \mathfrak{g}_V \times \mathfrak{g}_W \arrow[ll, "\text{adjoint quotient}"'] \arrow[d] \\
 & & (p_2)_! \Phi^c
\end{tikzcd}$$

$\mathcal{W}$ Whittaker perv. sh.

$c \in$ irr. number

$$(p_1)_! \mathcal{W} = (p_2)_! \Phi^c$$

raise Φ to the complex power c.

BDF [wrote $\mathcal{W}$ explicitly]