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MSRI

Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

# NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

**Name:** Mee Seong Im **Email/Phone:** mim2@illinois.edu

**Speaker's Name:** Nick Proudfoot

**Talk Title:** Quantizations of symplectic resolutions, Part 1.

**Date:** 9/4/14 **Time:** 3:30 am / pm (circle one)

**List 6-12 key words for the talk:** regular representation of the Weyl group, convolution operators, cohomology of the cotangent bundle of the flag variety, BGG category $\mathcal{O}$, Steinberg variety.

**Please summarize the lecture in 5 or fewer sentences:** Proudfoot will discuss the construction of the regular representation of the Weyl group via convolution operators on the cohomology of the cotangent bundle of the flag variety. He will then lift this action to a braid group action on BGG category $\mathcal{O}$. He will then generalize this picture via a different symplectic resolution.

# CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>Computer Presentations</u>: Obtain a copy of their presentation
  - <u>Overhead</u>: Obtain a copy or use the originals and scan them
  - <u>Blackboard</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - <u>Handouts</u>: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to <u>notes@msri.org</u> with the workshop name and your name in the subject line.

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# Quantizations of symplectic resolutions, part 1

Nick Proudfoot

Thurs, Sept 4, 2014, 3:30–4:30 pm

$G$ simple alg gp $/\mathbb{C}$

$B \subseteq G$ Borel

$X = G/B$

$$\begin{array}{l} M = T^*X = \{(gB, \sigma) \in X \times \mathrm{nil}(\mathfrak{g}) : g^{-1}\sigma g \in \mathfrak{b}\} \\ \downarrow \\ M_0 = \mathrm{nil}(\mathfrak{g}) \end{array}$$

Springer resolution

<u>EX</u>: $G = SL_2$

$$\begin{array}{l} M = T^*\mathbb{P}^1 \\ \downarrow \\ M_0 = \mathbb{C}^2/\{\pm 1\} \end{array}$$

$$Z = T^*\mathbb{P}^1 \times_{\mathbb{C}^2/\pm 1} T^*\mathbb{P}^1 = T^*\mathbb{P}^1_\Delta \cup_{\mathbb{P}^1} \mathbb{P}^1 \times \mathbb{P}^1$$

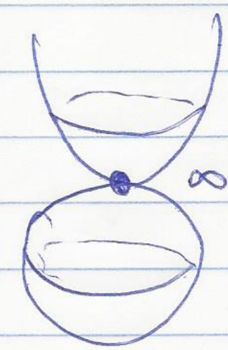
$Z = M \times_{M_0} M$ Steinberg variety

<u>Fact</u>: All components of $Z$ have dimension $d = \dim M$

$$M^+ = \bigcup \text{ conormal bdles to } B\text{-orbits in } X$$

Ex: $X = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

$$M^+ = \mathbb{C} \cup T^*_\infty \mathbb{CP}^1,$$




Consider $H^{BM}_{2d}(Z) =$

$$= \mathbb{C}^{\#\text{ of components of } Z}$$

This is an algebra:

$$\begin{tikzcd} & M \times_{M_0} M \times_{M_0} M \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] & \\ Z & Z & Z \end{tikzcd}$$

$$\alpha * \beta = (p_2)_*(p_1^* \alpha \cap p_3^* \beta)$$

$$H^{BM}_d(M^+) = \mathbb{C}^{\#\text{ of components of } M^+}$$

$\uparrow$ module over $H^{BM}_{2d}(Z)$

Thm (Ginzburg)

$$H_{2d}^{BM}(Z) \cong \mathbb{C}[W]$$

$$H_{d}^{BM}(M^+) \cong \mathbb{C}[W]$$

$$G \curvearrowright X \rightsquigarrow \mathfrak{g} \to \mathrm{vf}(X)$$

$$\rightsquigarrow U(\mathfrak{g}) \to \mathrm{Diff}(X) = \Gamma(X; D_X)$$

Fact: This map is surjective.

Let $U(\mathfrak{g})_0 = U(\mathfrak{g})/\ker \cong \mathrm{Diff}(X)$

Thm (Beilinson–Bernstein):

$$U(\mathfrak{g})_0\text{-mod} \underset{\Gamma}{\overset{\mathrm{Loc}}{\rightleftarrows}} D_X\text{-mod} \quad (\sim)$$

$$\begin{array}{ccccc} U(\mathfrak{g})_0\text{-mod} & \xrightarrow{\mathrm{Loc}} & D_X\text{-mod} & \overset{\text{"microlocal support"}}{\rightsquigarrow} & \text{Cycles on } M = T^*X \\ \cup\vert & & & & \cup\vert \\ \mathcal{O}_0 := \text{f.g. } U(\mathfrak{g})_0\text{-modules} & & \xrightarrow{\text{weakly } B\text{-equiv. } D_X\text{-mod.}} & & \text{Cycles on } M^+. \end{array}$$

that are locally finite for the action of $U(\mathfrak{b})$.

$$K(\mathcal{O}_0)_{\mathbb{C}} \xrightarrow{\ \cong\ } H_d^{BM}(M^+)$$

$U(\mathfrak{g})_0$-bimodules $\xrightarrow{Loc}$ $D_X \boxtimes D_X^{op}$-mod.

$\cup\!|$ $\qquad\qquad\qquad\qquad \wr$

$HC_0$ $\qquad$ cycles on $M \times M$

$\qquad\qquad\qquad\qquad \cup\!|$

$\| \xrightarrow[\text{G-eq. D-mods. on } X\times X]{}$ cycles on $Z$

f.g.

$U(\mathfrak{g})_0$-bimod.

that are loc. finite for the adj. action.

Thm ① $HC_0$ is a tensor cat. acting on $\mathcal{O}_0$,

② support intertwines $\overset{L}{\otimes}$ with $*$.

③ $\exists$ HC bimodules $\{H_w : w \in W\}$ s.t.

- $H_w \overset{L}{\otimes} - : D^b(\mathcal{O}_0) \xrightarrow[\Theta_w]{\sim} D^b(\mathcal{O}_0)$
- $\Theta_w \circ \Theta_{w'} \simeq \Theta_{ww'}$ if $\ell(w) + \ell(w') = \ell(ww')$

Thus $B_W \curvearrowright D^b(\mathcal{O}_0)$

categorifying $W \curvearrowright K(\mathcal{O}_0)_{\mathbb{C}}$.

Def. A conical symplectic resolution

is
- a smooth variety $M/\mathbb{C}$
- a symplectic form $\omega \in \Omega^2_{alg}(M)$
- commuting actions of $S = \mathbb{C}^\times$ and $T = \mathbb{C}^\times$

s.t.
- $T$ preserves $\omega$
- $S$ rotates $\omega$: $s^*\omega = s\omega \quad \forall s \in S$
- $S \circlearrowright \mathbb{C}[M]$ with nonnegative weights and $\mathbb{C}[M] = \mathbb{C}$.
- $|M^T| < \infty$.
- The map
$$\begin{array}{c} M \\ \downarrow \\ M_0 = \operatorname{Spec} \mathbb{C}[M] \end{array}$$
is a projective resolution.

Ex ①. $M = T^*(G/B)$ $\quad$ $S$ scales fibers
$M_0 = \operatorname{nil}(\mathfrak{g})$ $\quad$ $T \hookrightarrow G$ generic

② $M = \operatorname{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$
$$\downarrow$$
$M_0 = \operatorname{Sym}^n(\mathbb{C}^2/\Gamma)$,

$\Gamma$ abelian, $\Gamma = \mathbb{Z}_k$.

(3) $G \curvearrowright V$ linear action

$\chi : G \to \mathbb{C}^*$,

$G \curvearrowright T^*V \xrightarrow{\mu} \mathfrak{g}^*$

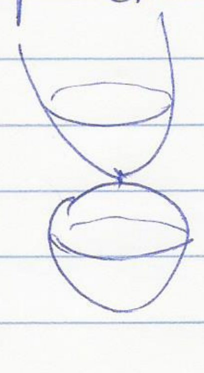
$$\begin{array}{c} M = \mu^{-1}(0) /\!\!/_{\chi}\, G \\ \downarrow \\ M_0 = \mu^{-1}(0) /\!\!/ G \end{array}$$

i.e., quiver varieties
Hypertoric varieties

Let $Z = M \times_{M_0} M$

$M^+ = \{ p \in M : \lim_{t \to 0} t \cdot p \text{ exists} \}$

If $M = T^*\mathbb{P}^1$



$= M^+$

$H_{2d}^{BM}(Z) \curvearrowright H_d^{BM}(M^+)$

alg. as before

**Def** A quantization of $(M, \omega)$ is

- a $T$-equiv. sheaf $\mathcal{A}$ of filtered algs on $M$
- a $T \times S$-graded isomorphism

$$\operatorname{gr} \mathcal{A} \cong \mathrm{Fun}_M .$$

Let $A = \Gamma(M, \mathcal{A})$

**Ex.** ① If $M = T^*(G/B)$, $A$ is quotient of $U(\mathfrak{g})$.

② If $M = \mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$, then $A$ is a quotient of a spherical rat'l Cherednik alg.

③ If $M = T^*V /\!\!/ G$, then $A$ is a quotient of $\mathrm{Diff}(V)^G$.

**Thm** (BPW):

$$A\text{-mod} \underset{\text{Loc.}}{\overset{\Gamma}{\rightleftarrows}} \mathcal{A}\text{-mod}$$

is an equiv. for "most" quantizations.

[MN: equiv. in abelian case]

[MN: equiv in derived case].

$$\begin{array}{ccccc} A\text{-mod} & \xrightarrow{\text{Loc}} & \mathcal{A}\text{-mod} & \xrightarrow{\text{Supp}} & \text{cycles on } M \\ \cup\!\!\!\mid & & & & \cup\!\!\!\mid \\ \mathcal{O} & & \longrightarrow & & \text{cycles on } M^+ \end{array}$$

$\mathcal{O}$ = fin. gen. $A$-mod that are loc. fin. for $A^+$ ← $T$-action.

**Thm** (BLPW): $K(\mathcal{O})_{\mathbb{C}} \xrightarrow{\sim} H_d^{BM}(M^+)$

$$\begin{array}{ccccc} A\text{-bimod} & \xrightarrow{Loc} & A \boxtimes A^{op} & \xrightarrow{supp} & \text{cycles on } M \times M \\ \cup\!\mid & & & & \cup\!\mid \\ HC & & \longrightarrow & & \text{cycles on } Z \end{array}$$

$HC$ = $N$ s.t. $gr\, N$ is supported on diag. of $M_0 \times M_0$.

**Thm** (BPW)

1. $HC$ is a tensor category acting on $\mathcal{O}$.
2. Supp intertwines $\overset{L}{\otimes}$ with $*$.
3. $\exists$ nice bimodules that fit together into a generalized braid group action on $D^b(\mathcal{O})$.