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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im **Email/Phone:** mim2@illinois.edu

Speaker's Name: Nick Proudfoot

Talk Title: Quantizations of symplectic resolutions, Part II

Date: 9/5/14 **Time:** 11:30 [✓ am] / pm (circle one)

List 6-12 key words for the talk: symplectic resolutions, dual pairs, quiver varieties, Satake equivalence.

Please summarize the lecture in 5 or fewer sentences: Proudfoot will give an introduction to the duality arising in symplectic resolutions, linking two famous geometric constructions of simple representations of algebraic groups via quiver varieties and the geometric Satake equivalence.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- Computer Presentations: Obtain a copy of their presentation
- Overhead: Obtain a copy or use the originals and scan them
- Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- Handouts: Obtain copies of and scan all handouts

☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Quantizations of symplectic resolutions, part II

Nick Proudfoot

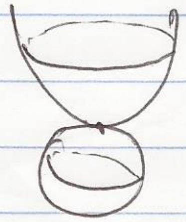
Friday, Sept 5, 2014, 11:30–12:30pm

$$\begin{tikzcd} M \arrow[d, "\substack{\text{conical}\\ \text{sympl.}\\ \text{resoln.}}"] & T^*\mathbb{P}^1 \arrow[d] \\ M_0 & \mathbb{C}^2/\{\pm 1\} \end{tikzcd}$$

$\mathbb{C}^* = T \curvearrowright M$

$M^+ = \{p \in M : \lim_{t \to 0} tp \text{ exists}\}$

$Z = M \times_{M_0} M$ e.g.



e.g. $T^*\mathbb{P}^1 \cup \mathbb{P}^1 \times \mathbb{P}^1$

A filt. alg. s.t. $\operatorname{gr} A \cong \mathbb{C}[M]$

e.g. $\Gamma(\mathbb{P}^1, D_{\mathbb{P}^1}) \cong U(sl_2)_0$.

$\mathcal{O}$ = some cat of A-modules

$$\begin{tikzcd} & & & \text{"big simples"} \\ \text{"little simples"} & \subseteq K(\mathcal{O})_{\mathbb{C}} \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & \supseteq & \mathbb{C}\{[L] : L \text{ simple}, \operatorname{Ann}(L)=0\} \arrow[dd, "\cong"] \\ \mathbb{C}\{[L] : L \text{ f.d. simple}\} \arrow[ddd, leftrightarrow] & H_d^{BM}(M^+) \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & \curvearrowleft & H_{2d}^{BM}(Z) \\ & H_T^d(M) & \supseteq & IH_T^d(M_0) \\ & & & \\ H^d(M) & & & \end{tikzcd}$$

Conjecture (BLPW):

① $\mathcal{O}$ is Koszul.

Ex i.) $T^*(G/B)$ (BGS)

ii.) $\lambda \geq \mu$ partitions of r

$\overline{N_\lambda} \supseteq N_\mu$

$X_{\lambda\mu}$ resolution of slice to N_μ inside $\overline{N_\lambda}$.

$\mathcal{O} \simeq$ singular block of parabolic $\mathfrak{sl}_r$ BGG, cat $\mathcal{O}$.

(Losev, Webster)

Koszul by BGS

iii.) $\mathcal{H}(k,r) = \mathrm{Hilb}^r(\mathbb{C}^2/\mathbb{Z}_k)$

$\mathcal{M}(k,r) = \left\{ \begin{array}{l} \text{tors. free sh. on } \mathbb{P}^2, \\ \text{framed at } \infty, \\ \text{rk } k, \ c_2 = r \end{array} \right\} / \sim$

Koszul: Chang–Miyachi, RSVV (general case)

iv.) Hypertoric varieties (BLPW)

② $\exists$

$$\begin{tikzcd} M^! \arrow[d] \\ M_0^! \end{tikzcd}$$

s.t. $\mathcal{O}^!$ is Koszul dual to $\mathcal{O}$.

Ex. i.) $T^*(G/B)$ dual to $T^*(G^\vee/B^\vee)$ (BGS)

ii.) $X_{\lambda\mu}$ is dual to $X_{\mu^t \lambda^t}$

(parabolic - singular duality)
(Bachelin)

iii.) $\mathcal{H}(k,r)$ dual to $\mathfrak{M}(k,r)$
(RSVV)

iv.) hypertoric vars. are dual to other hypertoric vars.
(BLPW)

③. $\mathcal{O}$ dual to $\mathcal{O}^! \Rightarrow \exists$ bij.

$\{$simples in $\mathcal{O}\} \longleftrightarrow \{$simples in $\mathcal{O}^!\}$

big simples $\longleftrightarrow$ little simples

little $\longleftrightarrow$ big.

$$\begin{tikzcd} M \arrow[d] & \text{is dual to} & M^! \arrow[d] \\ M_0 & & M_0^! \end{tikzcd}$$

Upshot: if the left is dual to the right,

then $H^d(M) \cong \mathbb{C}\{[L] : L \text{ little simple in } \mathcal{O}\}$

"looks like" $IH^{d^!}_T(M_0^!) \cong$

$\cong \mathbb{C}\{[L^!] : L^! \text{ big simple in } \mathcal{O}^!\}$

<u>Ex</u>:

$$\begin{tikzcd} M = T^*\mathbb{P}^2 = X_{\underset{\lambda}{\ydiagram{2,1}}\,\underset{\mu}{\ydiagram{1,1,1}}} = \mathcal{M}(3,1) \arrow[d] \\ M^{\circ} = \operatorname{spec} \mathbb{C}[T^*\mathbb{P}^2] = \left\{ \begin{array}{c} \text{nilpot. } 3\times 3 \\ \text{matrices} \\ \text{of rk} \leq 1 \end{array} \right\} \end{tikzcd}$$

$$\begin{tikzcd} X_{\ydiagram{3}, \ydiagram{2,1}} \arrow[d, phantom, "\shortparallel"] = \widetilde{\mathbb{C}^2/\mathbb{Z}_3} = M^! \arrow[d] \\ \mathcal{H}(3,1) \qquad \mathbb{C}^2/\mathbb{Z}_3 = M_0^! \end{tikzcd}$$

$\dim H^4(M) = 1$

$\dim IH^4_T(M_0) = 2$

$\dim H^2(M^!) = 2$

$\dim IH^2_T(M_0^!) = 1$

G simple alg. group of type ADE.

$H \subseteq G$ Cartan

$X = \mathrm{Hom}(H, \mathbb{C}^*)$

$\xi, \theta \in X$ dominant weights

$V(\xi)_\theta$ = θ-wt space of irrep with h.w. ξ

Nakajima:

$\alpha_1, \dots, \alpha_r \in X$ simple roots

$\Lambda_1, \dots, \Lambda_r \in X$ fundamental wts

$(\Lambda_i, \alpha_j) = \delta_{ij}$

Ex: $G = SL_{r+1}$

$X = \mathbb{Z}^{r+1}/\mathbb{Z}$

$\alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0)$

$\Lambda_i = (1, \dots, 1, 1, 0, \dots, 0)$

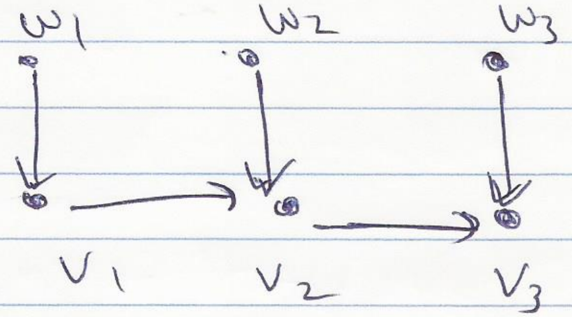
Define $v, w \in \mathbb{N}^r$ by

$\xi = \sum w_i \Lambda_i$

$\theta = \xi - \sum v_i \alpha_i$

Let Q be an ~~[illegible]~~ orientation of the Dynkin diagram of G.

$G = SL_4$



$$GL_V \curvearrowright \mathrm{Rep}(V,W) \curvearrowleft GL_W$$

$$\underset{\underset{GL_W}{\circlearrowleft}}{M(\xi,\theta)} := T^* \mathrm{Rep}(V,W) /\!\!/\!\!/ GL_V$$

Fact: if ξ is a sum of minuscule wts, then $\exists\, T \hookrightarrow GL_W$ s.t.

$$|M(\xi,\theta)^T| < \infty.$$

Thm (Nakajima)

$$\begin{tikzcd}
H^{BM}_{top}(M(\xi,\theta)) \arrow[r, phantom, "\cong"] & V(\xi)_\theta \arrow[dd, leftrightarrow, bend left=70, "\text{dual}"] \\
K(\mathcal{O})_{\mathbb{C}} \arrow[d, phantom, sloped, "\cong"] \arrow[r, phantom, "\cong"] & \mathbb{C}\{\text{little simples}\} \arrow[d, "\cong\ (\text{Bez--Losev})"] \\
H_T^{top}(M(\xi,\theta)) \arrow[r, phantom, "\cong"] & H^{top}(M(\xi,\theta))
\end{tikzcd}$$

$H^\vee \subseteq G^\vee$ Langlands dual to $H \subseteq G$

$$Gr := G^\vee((z)) / G^\vee[[z]]$$

affine Gr. for $G^\vee$.

$$\begin{tikzcd}
X = \mathrm{Hom}(H, \mathbb{C}^*) \arrow[r, phantom, "\cong"] & \mathrm{Hom}(\mathbb{C}^*, H^\vee) \arrow[d, phantom, "\subseteq" sloped] \\
& H^\vee((z)) \arrow[d, phantom, "\subseteq" sloped] \\
& G^\vee((z)) \arrow[d, two heads] \\
X \arrow[d, phantom, "\ni" sloped] & Gr \arrow[d, phantom, "\ni" sloped] \\
\xi \arrow[r, mapsto] & p_\xi
\end{tikzcd}$$

$$H^\vee \subseteq G^\vee \subseteq G^\vee[[z]] \curvearrowright Gr.$$

<u>Ex</u>: $Gr^{H^\vee} = \{p_\xi : \xi \in X\}$

Let $\mathcal{O}_\xi := G^\vee[[z]] \cdot p_\xi$

<u>Thm</u> (Ginzburg, MV):

$$\left\{ \begin{array}{l} G^\vee[[z]]\text{-equ.} \\ \text{perv. sh. on} \\ Gr \text{ w/ f.d. support} \end{array} \right\} \longleftrightarrow \mathrm{Rep}(G)$$

$$IC(\mathcal{O}_\xi) \longleftrightarrow V(\xi) \quad \text{simple rep w/ h.w. } \xi.$$

H^* ⟷ forget the G-action.

$$\Rightarrow \quad IH(\overline{\mathcal{O}}_\xi) \cong \underline{V(\xi)}$$

Ginzburg: can choose $T \hookrightarrow H^\vee$ and tweak the equiv. to get

$$IH_T^{top}(\overline{\mathcal{O}}_\xi) \cong \underline{V(\xi)}.$$

$$V(\xi) \cong IH_T^{top}(\overline{\mathcal{O}}_\xi)$$

$$\bigoplus_\theta V(\xi)_\theta \underset{loc.}{\cong} \bigoplus_\theta H_T^{top}(i_\theta^! IC(\mathcal{O}_\xi))$$

$$\bigoplus_\theta V(\xi)_\theta \cong \bigoplus_\theta IH_T^{top}(\text{slice to } \mathcal{O}_\theta \text{ in } \overline{\mathcal{O}}_\xi)$$

__Fact__ If ξ is a sum of minuscule wts, $\exists$

$$\begin{tikzcd} S(\xi,\theta) \arrow[d, "\text{conical symp. res.}"] \\ \text{slice} \end{tikzcd}$$

__Conj.__ (BLPW):

$M(\xi,\theta)$ dual to $S(\xi,\theta)$

So
$$\begin{tikzcd}
H^{top}(M(\xi,\theta)) \arrow[r, leftrightarrow, "dual"] \arrow[d, leftrightarrow, "dual"'] & IH_T^{top}(slice) \arrow[d, phantom, "\wr\wr\ GS"] \\
H_{top}(M(\xi,\theta)) \arrow[r, phantom, "\cong_{Nak}"] & V(\xi)_\theta
\end{tikzcd}$$

In type A, $\exists$ partitions λ, μ s.t.

$$M(\xi,\theta) \cong X_{\lambda\mu} \text{ (Maffei).}$$

$$S(\xi,\theta) \cong X_{\mu^t\lambda^t} \text{ (Milovic-Vybornov)}$$