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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im **Email/Phone:** mim2@illinois.edu

Speaker's Name: Olivier Schiffmann

Talk Title: Quivers, curves, Kac polynomials and the number of stable Higgs bundles

Date: 9/5/14 **Time:** 2:00 am **pm** (circle one)

List 6-12 key words for the talk: quivers, Kac polynomials, indecomposable vector bundles, stable Higgs bundles

Please summarize the lecture in 5 or fewer sentences: Schiffmann replaces the category of representations of a quiver by the category of coherent sheaves on a smooth proj curve, explaining a "global" analog of a certain conjecture by Kac. As an application, Schiffmann gives a formula for the number of stable Higgs bundles over a sm proj curve over a finite field.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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Quivers, curves, Kac poly's and the number of stable Higgs bundles
Olivier Schiffmann
Fri, Sept 5, 2014, 2-3 pm

§5. Counting stable Higgs bundles.

__Fix__ r, d s.t. $\gcd(r,d) = 1$
$\Rightarrow$ $\mathrm{Higgs}^{st}_{r,d} = \mathrm{Higgs}^{sst}_{r,d}$
$\Rightarrow$ indecomp $\Rightarrow$ abs. indecomp.

__Aim__: show $\# \mathrm{Higgs}^{st}_{r,d}(\mathbb{F}_q) = q^{1+(g-1)r^2} A_{r,d}(X)$

First idea,

$$\mathrm{Higgs}^{st}_{r,d} \longrightarrow \mathrm{Bun}_{r,d}$$

lands into $\mathrm{Indec}_{r,d}$

$$\cap|$$

$$\mathrm{Bun}_{r,d}.$$

__Quivers__: Crawley-Boevey, Van den Bergh
Nakajima

__Key idea__:

$$\begin{tikzcd} & \mathrm{Bun}_{r,d} & \\ \mathrm{Higgs}^{st}_{r,d} \arrow[ur] \arrow[rr, hook] \arrow[d] & & Y \arrow[ul] \\ \{0\} \arrow[rr, hook] & & \mathbb{A}^1 \end{tikzcd}$$

$\mathbb{F}_q$-pts are the same for all fibers.

For $t \neq 0$, $p|_{Y_t} \longrightarrow \mathrm{Bun}_{r,d}$ is a "affine fibration" over $\mathrm{Indecom}_{r,d}$.

Modern 21^{st} Century mathematician's way to construct Y
$\supseteq$ { derived cotangent stalk }

My method: construct it explicitly using local presentation of $Higgs_{r,d}$.

Explicit construction of $Bun_{r,d}$ via Quot schemes
(ref: Le Potier)

Fix $(\mathcal{L}_1, \mathcal{L}_2) \in Pic^{-n} \times Pic^{-m}$, $n, m \gg 0$.

$\alpha := (r, d)$, $G = GL(V_1) \times GL(V_2)$
$l_i = \langle \mathcal{L}_i, \alpha \rangle$ Euler form
$V_i = k^{l_i}$.

$$Quot^{\alpha}_{\mathcal{L}_1, \mathcal{L}_2} = \{ \varphi : \bigoplus \mathcal{L}_i \otimes V_i \twoheadrightarrow \mathcal{F} : \mathcal{F} \text{ coh. sheaf of class } \alpha \} / \sim$$

$\varphi \sim \varphi'$ if $\ker \varphi = \ker \varphi'$.

<u>Thm</u> (Grothendieck)

$Quot^{\alpha}_{\mathcal{L}_1, \mathcal{L}_2}$ is a proj. G-alg. variety.

$$Q^\alpha = \{ \varphi : \bigoplus \mathcal{L}_i \otimes V_i \twoheadrightarrow \mathcal{F} : \mathcal{F} \text{ bundle of class } \alpha,\ \varphi_* : V_i \xrightarrow{\sim} \mathrm{Hom}(\mathcal{L}_i, \mathcal{F})\ \forall i \}$$

$\bigcap |$ open, smooth

$\mathrm{Quot}^\alpha_{\mathcal{L}_1, \mathcal{L}_2}$

$$\mathrm{Bun}_\alpha^{\mathcal{L}_1, \mathcal{L}_2} = \left[Q^\alpha / G \right]$$ open substack of Bun_α

$$\mathrm{Bun}_\alpha = \bigcup_{\mathcal{L}_1, \mathcal{L}_2} \mathrm{Bun}_\alpha^{\mathcal{L}_1, \mathcal{L}_2}$$

$$T^* Q^\alpha = \{ (\varphi, \theta) : \varphi : \bigoplus \mathcal{L}_i \otimes V_i \twoheadrightarrow \mathcal{F} \in Q^\alpha,\ \theta \in \mathrm{Hom}(\ker \varphi, \mathcal{F})^* \}$$

$$(T_\varphi Q^\alpha = \mathrm{Hom}(\ker \varphi, \mathcal{F}))$$

Fundamental exact sequence

$$0 \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})^* \to \mathrm{Hom}(\ker \varphi, \mathcal{F})^* \xrightarrow[\mu]{} \underset{\mathfrak{g}^*}{\underset{\wr\!\wr\,\varphi}{\mathrm{Hom}(\bigoplus \mathcal{L}_i \otimes V_i, \mathcal{F})^*}} \to \mathrm{End}(\mathcal{F})^*$$

$[\mu^{-1}(0)/G] \underset{\text{open}}{\subset} \mathrm{Higgs}_{r,d}$

For $n, m \gg 0$,

$$\mu^{-1}(0)^{st}/G = \mathrm{Higgs}^{st}_{r,d}$$

Idea: Consider other fibers of μ
Choose a line $L \cong \mathbb{A}^1 \hookrightarrow (\mathfrak{g}^*)^G$
and put $Y = \mu^{-1}(L)^*/G$.

<u>Fact</u> (easy) $\gcd(r,d)=1 \rightsquigarrow$ may choose $\mathcal{L}_1, \mathcal{L}_2$ s.t. $\gcd(l_1, l_2)=1$.

$$L = k\lambda \quad \text{with} \quad \lambda(u_1,u_2) = l_2\, tr(u_1) = l_1\, tr(u_2)$$

$$\Rightarrow \begin{cases} \lambda(Id, Id) = 0 & \\ \lambda(e_1, e_2) \neq 0 & \forall \text{ proper projectors } e_1, e_2. \end{cases}$$

<u>Lemma</u> Let $\varphi: \bigoplus \mathcal{X}_i \otimes V_i \twoheadrightarrow \mathcal{F}$.
Then $L \subseteq \mathrm{Im}\, \mu_\varphi$ iff $\mathcal{F}$ is indecom.

<u>Sketch of Proof</u>
i.) $\mathcal{F}$ indecomp. $\Leftrightarrow$ $End(\mathcal{F})$ is a local alg., i.e., $\forall f \in End(\mathcal{F})$
$f = c \cdot Id + \text{nilp}$.

ii.) $L \subseteq \mathrm{Im}(\mu_\varphi) \Leftrightarrow \lambda(f \circ \varphi) = 0$
$\forall f \in End(\mathcal{F})$.

Put $\mathcal{X} = \mu^{-1}(L)$

$\mathcal{X}_t = \mu^{-1}(\{t\lambda\})$

$$p : [T^* Q_\alpha / G] \longrightarrow \mathrm{Bun}_{r,d}$$

<u>Cor.</u> $\forall\, t \neq 0$, $\mathrm{Im}(p_t) = \mathrm{Indecomp}_{r,d}$ and all fibers have cardinality $q^{1+(g-1)r^2}$.

<u>Lemma</u> $\forall\, t \neq 0$, $\mathcal{X}_t \subseteq \mathcal{X}^{st}$.
(follows from genericity of λ).

<u>Sum up</u>

$$\begin{tikzcd} Y_0 = \mathrm{Higgs}^{st} \arrow[r, hook] \arrow[d] & Y \arrow[d] & Y' \arrow[l, hook'] \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbb{A}^1 & \mathbb{A}^1 \setminus \{0\} \arrow[l, hook'] \end{tikzcd}$$

smooth, Y_t smooth.

$$\# Y_t(\mathbb{F}_q) = q^{1+(g-1)r^2} A_{r,d}(X)$$

Enough to show:

$$\# Y_0(\mathbb{F}_q) = \# Y_t(\mathbb{F}_q)$$

$$q \# Y_0(\mathbb{F}_q) = \# Y(\mathbb{F}_q)$$

$\exists\ \mathbb{G}_m$-action on Y,
$$z \cdot (\varphi, \theta) \mapsto (\varphi, z\cdot\theta)$$

Key pt: this action is contracting

$\Rightarrow Y^{\mathbb{G}_m} = \coprod Z_i$, Z_i sm, proj. varieties
$Z_i \subseteq Y_0$.

BB decomp
$$Y = \coprod Z_i^+, \quad Z_i^+ \xrightarrow[\dim = d_i]{\text{affine}} Z_i$$
$$Y_0 = \coprod Z_{i,0}^+ \quad Z_{i,0}^+ \xrightarrow[\dim = d_{i,0}]{\text{affine}} Z_i$$

$\rightsquigarrow \quad d_i = d_{i,0} + 1.$

Proof of key pt

$Y_0 \xrightarrow{\mu} \mathbb{A}$ Hitchin map,
(proper)
$\mathbb{G}_m$-equiv. +
$\mathbb{G}_m$-action on $\mathbb{A}$
is contracting.

Y construct an avatar of Hitchin map
(Alvarez-Consul, King).

2nd method) (Mozgovoy–S)

Use Hall alg. techniques
(HN filtration, ≥ 0 truncation, Jordan decomp)
For cat. of (twisted) Higgs bdles + cohom. (Ext^2) vanishing

$$\left(1_{\mathrm{Higgs}_{r,d}},\ 1_{\mathrm{Higgs}^{st}_{\alpha_1}}, \dots, 1_{\mathrm{Higgs}^{st}_{\alpha_r}}\right)$$

$$\overset{\uparrow}{=} \left(\Delta(1_{\mathrm{Higgs}_{r,d}}),\ 1_{\mathrm{Higgs}^{st}_{\alpha_1}}, \dots, 1_{\mathrm{Higgs}^{st}_{\alpha_r}}\right)$$

Hopf pairing is dual

Define $A_{g,r}(z)$ by

$$\sum_{r\geq 1} A_{g,r}(z)\, T^r = (q-1) \operatorname{Log}\Big(\sum_{\lambda} q^{(g-1)\langle\lambda,\lambda\rangle} \cdot J_\lambda(z)\, H_\lambda(z)\, T^{|\lambda|}\Big).$$

Then for any $d \in \mathbb{Z}$, have

$$A_{g,r,d} = -\sum_{\xi\in\mu_r} \xi^{-d}\, \mathrm{Res}_{z=\xi}\Big(A_{g,r}(z)\, \frac{dz}{z}\Big),$$

where μ_r = set of r^{th} roots of unity.

[arXiv: 1406.3839]

§ 6 ...

- Can compute $\# Nil^{st}_X(\mathbb{F}_q)$ from $\# Higgs^{st}_X(\mathbb{F}_q)$
  $\rightsquigarrow$ compute $\# Irr(Nil^{st}_X)$.

- $Higgs^{st}_{r,d}$ pure
  $\rightsquigarrow$ can compute $H^*(Higg^{st}_{r,d})$
  for $X/\mathbb{C}$
  $X/\mathbb{F}_q$

$$H^*(Higg^{st}_{r,d}) \overset{\text{diffeom. when } X/\mathbb{C}}{\cong} H^*(M^{tw}_{g,r})$$

"twisted" character variety (rep of $\pi_1(\Sigma_g)$)

- $H^*(Higgs^{st}_{r,d})$ $r=2$: Hitchin
  $r=3$: Gothen
  2006: Hausel, Rodriguez-Villegas
  Conj for MH poly of $M^{tw}_{g,r}$
  $\Rightarrow$ Conj. for $H^*(M^{tw}_{g,r}) = H^*(Higgs^{st}_{r,d})$

$$(H^*(M^{tw}_{g,r}), \text{weight}) \xleftrightarrow{\text{conj.}} (H^*(Higgs^{st}_{r,d}), \text{perverse})$$

relate relate

$$\begin{pmatrix} \# A_{g,r,d} \\ \text{HN filtration} \end{pmatrix}$$

∘ Analogy with Kac's conj. for quivers

$A_d(z) \in \mathbb{N}(z)$

$A_d(0) = \dim \mathfrak{g}_{\vec{Q}, d}$

No good analogy for $\mathfrak{g}_{\vec{Q}}$ for curves

$\exists$ " " " $U_q^+(\mathfrak{g}_{\vec{Q}})$ " ",

namely, $\mathcal{H}_X^{sph}$.

Expect $A_d(0) \sim \# \mathrm{Irr}(\mathrm{Nil}_{r,d}^{st})$

$\sim \underbrace{\text{nbm of gens}}_{}$ of $\mathcal{H}_X^{sph}$.

DT-theory

for g-loop quivers +

for the moduli stack of

Higgs bundles.