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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Mee Seong Im **Email/Phone:** Mim2@illinois.edu

Speaker's Name: Paul Baum

Talk Title: Representations of p-adic groups

Date: 9/5/14 **Time:** 3:30 am / **pm** (circle one)

List 6-12 key words for the talk: geometric structure, local Langlands Conjecture, Hecke algebra, cuspidal pair, springer correspondence (slide 45/53).

Please summarize the lecture in 5 or fewer sentences: We build some background to relate $T_\alpha // \Gamma_\alpha$ and $\hat{G}_\alpha$ where T_α is Bernstein's complex torus (a finite quotient of the complex torus consisting of all unramified characters of M) and Γ_α is the subgroup of the Weyl group of M s.t. $\exists$ an unramified character χ of M with $w \ast \sigma \sim \chi \otimes \sigma$ (slide 26/53) and $\hat{G}_\alpha$ is the set of all irreducible subquotients of $\mathrm{Ind}_M^G(\chi \otimes \sigma)$; $\hat{G}_\alpha$ is also called the Bernstein component.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
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Geometric Structure and the Local Langlands Conjecture

Paul Baum
Penn State

MSRI
Geometric Representation Theory

September 5, 2014

GEOMETRIC STRUCTURE AND THE LOCAL LANGLANDS CONJECTURE

Let G be a reductive p-adic group which is *connected* and split. Examples are $GL(n,F), SL(n,F), SO(n,F), Sp(2n,F), PGL(n,F)$ where n can be any positive integer and F can be any finite extension of the field Q_p of p-adic numbers. The smooth (or admissible) dual of G is the set of equivalence classes of smooth irreducible representations of G. Within the smooth dual there are subsets known as the Bernstein components, and the smooth dual is the disjoint union of the Bernstein components. This talk will explain a conjecture due to Aubert-Baum-Plymen-Solleveld (ABPS) which says that each Bernstein component is a complex affine variety. These affine varieties are explicitly identified as certain extended quotients.

Joint work with Anne-Marie Aubert, Roger Plymen, and Maarten Solleveld.

Reference.

Geometric structure in smooth dual and the local Langlands conjecture (with A. M. Aubert, R. J. Plymen, and M. Solleveld — expository paper based on the Takagi lectures given by P. F. Baum at the November 2012 meeting of the Mathematical Society of Japan) Japanese Journal of Mathematics 9, 1-38, 2014.

Equivalence of categories

$$\left(\begin{array}{c} \text{Commutative unital finitely generated} \\ \text{nilpotent} - \text{free } \ \mathbb{C} \ \text{ algebras} \end{array} \right) \cong \left(\begin{array}{c} \text{Affine algebraic} \\ \text{varieties over } \mathbb{C} \end{array} \right)^{op}$$

$$\mathcal{O}(X) \longleftarrow X$$

The extended quotient

Let Γ be a finite group acting on an affine variety X.

X is an affine variety over the complex numbers $\mathbb{C}$.

$$\Gamma \times X \longrightarrow X$$

The quotient variety X/Γ is obtained by collapsing each orbit to a point.

For $x \in X$, Γ_x denotes the stabilizer group of x.

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}$$

$c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x.

The extended quotient is obtained by replacing the orbit of x by $c(\Gamma_x)$.

This is done as follows:

Set $\widetilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$

$\widetilde{X} \subset \Gamma \times X$

$\widetilde{X}$ is an affine variety and is a sub-variety of $\Gamma \times X$.

Γ acts on $\widetilde{X}$.

$\Gamma \times \widetilde{X} \to \widetilde{X}$

$g(\gamma, x) = (g\gamma g^{-1}, gx) \qquad g \in \Gamma \qquad (\gamma, x) \in \widetilde{X}$

The extended quotient, denoted $X//\Gamma$, is $\widetilde{X}/\Gamma$.

i.e. The extended quotient $X//\Gamma$ is the ordinary quotient for the action of Γ on $\widetilde{X}$.

The extended quotient is an affine variety.

$\widetilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$

The projection $\widetilde{X} \to X$

$$(\gamma, x) \mapsto x$$

is Γ-equivariant and, therefore, passes to quotient spaces to give a map

$$\rho : X//\Gamma \to X/\Gamma$$

ρ is the projection of the extended quotient onto the ordinary quotient.

$X/\Gamma \hookrightarrow X//\Gamma \to X/\Gamma$

$x \mapsto (e, x)$
e=identity element of Γ.

$X/\Gamma \hookrightarrow X//\Gamma$ is the inclusion of the ordinary quotient in the extended quotient.

Since G — in the topology it receives from F — is locally compact we may fix a (left-invariant) Haar measure dg for G.

The Hecke algebra of G, denoted $\mathcal{H}G$, is then the convolution algebra of all locally-constant compactly-supported complex-valued functions $f : G \to \mathbb{C}$.

$$\begin{aligned}(f+h)(g) &= f(g)+h(g) \\ (f*h)(g_0) &= \int_G f(g)h(g^{-1}g_0)dg\end{aligned} \quad \left\{\begin{array}{l} g \in G \\ g_0 \in G \\ f \in \mathcal{H}G \\ h \in \mathcal{H}G \end{array}\right.$$

Definition

A *representation* of the Hecke algebra $\mathcal{H}G$ is a homomorphism of $\mathbb{C}$ algebras

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

where V is a vector space over the complex numbers $\mathbb{C}$.

Definition

A representation

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

of the Hecke algebra $\mathcal{H}G$ is *irreducible* if $\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$ is not the zero map and $\nexists$ a vector subspace W of V such that W is preserved by the action of $\mathcal{H}G$ and $\{0\} \neq W$ and $W \neq V$.

Definition

A *primitive ideal* I in $\mathcal{H}G$ is the null space of an irreducible representation of $\mathcal{H}G$.

Thus

$$0 \longrightarrow I \hookrightarrow \mathcal{H}G \xrightarrow{\psi} \mathrm{End}_{\mathbb{C}}(V)$$

is exact where ψ is an irreducible representation of $\mathcal{H}G$.

There is a (canonical) bijection of sets

$$\widehat{G} \longleftrightarrow \mathrm{Prim}(\mathcal{H}G)$$

where $\mathrm{Prim}(\mathcal{H}G)$ is the set of primitive ideals in $\mathcal{H}G$.

Bijection (of sets)

$$\widehat{G} \longleftrightarrow \mathrm{Prim}(\mathcal{H}G)$$

What has been gained from this bijection?

On $\mathrm{Prim}(\mathcal{H}G)$ have a topology — the Jacobson topology.

If S is a subset of $\mathrm{Prim}(\mathcal{H}G)$ then the closure $\overline{S}$ (in the Jacobson toplogy) of S is

$$\overline{S} = \{J \in \mathrm{Prim}(\mathcal{H}G) \ | \ J \supset \bigcap_{I \in S} I\}$$

$\mathrm{Prim}(\mathcal{H}G)$ (with the Jacobson topology) is the disjoint union of its connected components.

Point set topology. In a topological space W, a subset A is connected iff whenever U_1, U_2 are two open sets of W with $A \subset U_1 \cup U_2$ and $U_1 \cap A \neq \emptyset$ and $U_2 \cap A \neq \emptyset$ then $A \cap U_1 \cap U_2 \neq \emptyset$.

Two points w_1, w_2 of W are in the same connected component if and only if $\exists$ a connected subset A of W with $w_1 \in A$ and $w_2 \in A$.

As a set, W is the disjoint union of its connected components. If each connected component is both open and closed, then as a topological space W is the disjoint union of its connected components.

$\widehat{G} = \mathrm{Prim}(\mathcal{H}G)$ (with the Jacobson topology) is the disjoint union of its connected components. Each connected component is both open and closed. The connected components of $\widehat{G} = \mathrm{Prim}(\mathcal{H}G)$ are known as the *Bernstein components*.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ denotes the set of connected components of $\mathrm{Prim}(\mathcal{H}G)$.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ is a countable set and has no further structure.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ is the *Bernstein spectrum* of G.

$\pi_o\mathrm{Prim}(\mathcal{H}G) = \{(M,\sigma)\}/\sim$ where (M,σ) can be any cuspidal pair i.e.
M is a Levi factor of a parabolic subgroup P of G
and σ is an irreducible super-cuspidal representation of M.

$\sim$ is the conjugation action of G, combined with tensoring σ by unramified characters of M.

"unramified" $=$ "the character is trivial on every compact subgroup of M."

$\pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$
$(M, \sigma) \sim (M', \sigma')$ iff there exists an unramified character
$\psi\colon M \to \mathbb{C}^\times = \mathbb{C} - \{0\}$ of M and an element g of G, $g \in G$, with

$$g(M, \psi \otimes \sigma) = (M', \sigma')$$

The meaning of this equality is:

- $gMg^{-1} = M'$
- $g_*(\psi \otimes \sigma)$ and σ' are equivalent
  smooth irreducible representations of M'.

For each $\alpha \in \pi_o\mathrm{Prim}(\mathcal{H}G)$,
$\widehat{G}_\alpha$ denotes the connected component of $\mathrm{Prim}(\mathcal{H}G) = \widehat{G}$.

The problem of describing $\widehat{G}$ now breaks up into two problems.

Problem 1 Describe the Bernstein spectrum
$\pi_o\mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$.

Problem 2 For each $\alpha \in \pi_o\mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$,
describe the Bernstein component $\widehat{G}_\alpha$.

Problem 1 involves describing the irreducible super-cuspidal representations of Levi subgroups of G. The basic conjecture on this issue is that if M is a reductive p-adic group (e.g. M is a Levi factor of a parabolic subgroup of G) then any irreducible super-cuspidal representation of M is obtained by smooth induction from an irreducible representation of a subgroup of M which is compact modulo the center of M. This basic conjecture is now known to be true in many examples.

For Problem 2, the ABPS conjecture proposes that each Bernstein component $\widehat{G}_\alpha$ has a very simple geometric structure.

Notation

$\mathbb{C}^\times$ denotes the (complex) affine variety $\mathbb{C} - \{0\}$.

Definition

A *complex torus* is a (complex) affine variety T such that there exists an isomorphism of affine varieties

$$T \cong \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times.$$

Bernstein assigns to each $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$ a complex torus T_α and a finite group Γ_α acting on T_α.

T_α is a complex algebraic group and $\exists$ a non-negative integer r such that T_α as an algebraic group defined over $\mathbb{C}$ is (non-canonically) isomorphic to $(\mathbb{C}^\times)^r := \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$. $\quad \mathbb{C}^\times := \mathbb{C} - \{0\}$

$$T_\alpha \cong \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$$

In general, Γ_α acts on T_α not as automorphisms of the algebraic group T_α but only as automorphisms of the underlying complex affine variety T_α.

Bernstein then forms the quotient variety T_α/Γ_α and proves that there is a surjective map π_α mapping $\widehat{G}_\alpha$ onto T_α/Γ_α .

$$\begin{array}{c} \widehat{G}_\alpha \\ \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha \end{array}$$

This map π_α is referred to as the infinitesimal character or the central character or the cuspidal support map.

$$\begin{array}{c} \widehat{G}_\alpha \\ \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha \end{array}$$

π_α is surjective, finite-to-one and generically one-to-one.

$\pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$

Given a cuspidal pair (M, σ), let $W_G(M)$ be the Weyl group of M.

$$W_G(M) := N_G(M)/M$$

Bernstein's finite group Γ_α is the subgroup of $W_G(M)$:

$$\Gamma_\alpha := \{w \in W_G(M) \,|\, \exists \, \text{an unramified character}\, \chi \,\text{of}\, M \,\text{with}\, w_*\sigma \sim \chi \otimes \sigma\}$$

Bernstein's complex torus T_α is a finite quotient of the complex torus consisting of all unramified characters of M.

$\pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$

Given a cuspidal pair (M, σ), the Bernstein component $\widehat{G}_\alpha \subset \widehat{G}$ consists of all irreducible sub-quotients of $Ind_M^G(\chi \otimes \sigma)$ where Ind_M^G is (smooth) parabolic induction and χ ranges over all the unramified characters of M.

$$\begin{array}{c} \widehat{G}_\alpha \\ \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha \end{array}$$

π_α is surjective, finite-to-one and generically one-to-one.

Conjecture

Let G be a connected split reductive p-adic group.
Let $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$.
Then there is a certain resemblance between

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & & \widehat{G}_\alpha \\ \rho_\alpha \Big\downarrow & \text{and} & \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha \end{array}$$

Conjecture

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & & \widehat{G}_\alpha \\ \rho_\alpha \big\downarrow & \text{and} & \big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha \end{array}$$

are almost the same.

How can this conjecture be made precise?
What does "almost the same" mean?
Let G be a connected split reductive p-adic group.
Let $\widehat{G}_\alpha$ be a Bernstein component in the smooth dual of G.

Let G be a connected split reductive p-adic group.
Let $\widehat{G}_\alpha$ be any Bernstein component in $\widehat{G}$.

Conjecture

There exists a bijection

$$\nu_\alpha \colon T_\alpha /\!/ \Gamma_\alpha \longleftrightarrow \widehat{G}_\alpha$$

with the following properties.

$\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$
Within the admissible dual $\widehat{G}$ have the tempered dual $\widehat{G}_{tempered}$.
$\widehat{G}_{tempered} =$ {smooth tempered irreducible representations of $G\}/\sim$
$\widehat{G}_{tempered} =$ Support of the Plancherel measure
$K_\alpha =$ maximal compact subgroup of T_α.
K_α is a compact torus. The action of Γ_α on T_α preserves the maximal compact subgroup K_α , so can form the compact orbifold $K_\alpha // \Gamma_\alpha$.

Conjecture : Properties of the bijection ν_α

- The bijection $\nu_\alpha : T_\alpha // \Gamma_\alpha \longleftrightarrow \widehat{G}_\alpha$ maps $K_\alpha // \Gamma_\alpha$ onto $\widehat{G}_\alpha \cap \widehat{G}_{tempered}$
  $K_\alpha // \Gamma_\alpha \longleftrightarrow \widehat{G}_\alpha \cap \widehat{G}_{tempered}$

Conjecture : Properties of the bijection ν_α

- For many α the diagram

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & \xrightarrow{\nu_\alpha} & \widehat{G}_\alpha \\ \rho_\alpha \downarrow & & \downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha & \xrightarrow[I]{} & T_\alpha/\Gamma_\alpha \end{array}$$

does not commute.
$I =$ the identity map of T_α/Γ_α.

Conjecture : Properties of the bijection ν_α

- In the possibly non-commutative diagram

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & \xrightarrow{\nu_\alpha} & \widehat{G}_\alpha \\ \Big\downarrow \rho_\alpha & & \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha & \xrightarrow[I]{} & T_\alpha/\Gamma_\alpha \end{array}$$

the bijection $\nu_\alpha : T_\alpha//\Gamma_\alpha \longrightarrow \widehat{G}_\alpha$ is continuous where $T_\alpha//\Gamma_\alpha$ has the Zariski topology and $\widehat{G}_\alpha$ has the Jacobson topology
AND the composition

$$\pi_\alpha \circ \nu_\alpha : T_\alpha//\Gamma_\alpha \longrightarrow T_\alpha/\Gamma_\alpha$$

is a morphism of affine algebraic varieties.

Conjecture : Properties of the bijection ν_α

- For each $\alpha \in \pi_o\mathrm{Prim}(\mathcal{H}G)$ there is an algebraic family

$$\theta_t : T_\alpha//\Gamma_\alpha \longrightarrow T_\alpha/\Gamma_\alpha$$

of morphisms of algebraic varieties, with $t \in \mathbb{C}^\times$, such that

$$\theta_1 = \rho_\alpha \qquad \text{and} \qquad \theta_{\sqrt{q}} = \pi_\alpha \circ \nu_\alpha$$

$\mathbb{C}^\times = \mathbb{C} - \{0\}$
q = order of the residue field of the p-adic field F over which G is defined
π_α = infinitesimal character of Bernstein

Conjecture : Properties of the bijection ν_α

- Fix $\alpha \in \pi_o\mathrm{Prim}(\mathcal{H}G)$. For each irreducible component $Z \subset T_\alpha//\Gamma_\alpha$ (Z is an irreducible component of the affine variety $T_\alpha//\Gamma_\alpha$) there is a cocharacter

$$h_Z : \mathbb{C}^\times \longrightarrow T_\alpha$$

such that

$$\theta_t(x) = \lambda(h_Z(t) \cdot x)$$

for all $x \in Z$.

cocharacter = homomorphism of algebraic groups $\mathbb{C}^\times \longrightarrow T_\alpha$
$\lambda : T_\alpha \longrightarrow T_\alpha/\Gamma_\alpha$ is the usual quotient map from T_α to T_α/Γ_α.

Question

Where are these correcting co-characters coming from?

Answer

In examples, the correcting co-characters are produced by the $SL(2,\mathbb{C})$ part of the Langlands parameters.

$$\mathcal{W}_F \times SL(2,\mathbb{C}) \longrightarrow {}^{L}G$$

Example

$G = GL(2, F)$
F can be any finite extension of the p-adic numbers $\mathbb{Q}_p$.
q denotes the order of the residue field of F.
$\widehat{G}_\alpha = \{$ Smooth irreducible representations of $GL(2, F)$ having a non-zero Iwahori fixed vector$\}$

$$T_\alpha = \{\text{unramified characters of the maximal torus of } GL(2,F)\} = \mathbb{C}^\times \times \mathbb{C}^\times$$

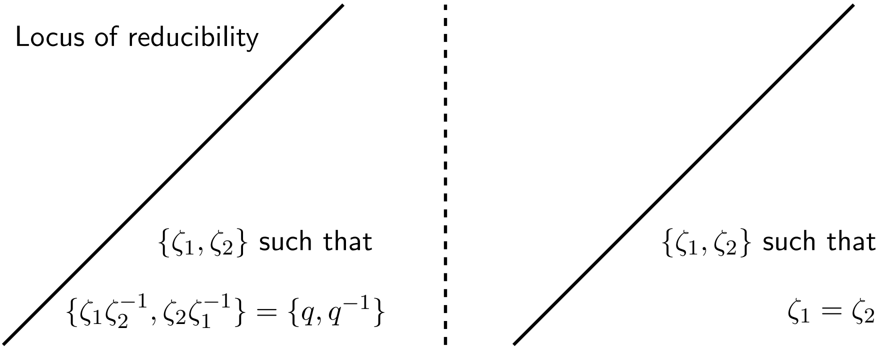
$\Gamma_\alpha =$ the Weyl group of $GL(2, F) = \mathbb{Z}/2\mathbb{Z}$

$0 \neq \gamma \in \mathbb{Z}/2\mathbb{Z} \qquad \gamma(\zeta_1, \zeta_2) = (\zeta_2, \zeta_1) \qquad (\zeta_1, \zeta_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$

$$(\mathbb{C}^\times \times \mathbb{C}^\times)//(\mathbb{Z}/2\mathbb{Z}) = (\mathbb{C}^\times \times \mathbb{C}^\times)/(\mathbb{Z}/2\mathbb{Z}) \bigsqcup \mathbb{C}^\times$$

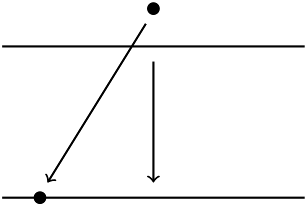
$$(\mathbb{C}^\times \times \mathbb{C}^\times)//(\mathbb{Z}/2\mathbb{Z}) = (\mathbb{C}^\times \times \mathbb{C}^\times)/(\mathbb{Z}/2\mathbb{Z}) \coprod \mathbb{C}^\times$$

$$(\mathbb{C}^\times \times \mathbb{C}^\times)/(\mathbb{Z}/2\mathbb{Z})$$

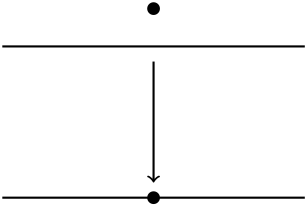


correcting cocharacter $\mathbb{C}^\times \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times$ is $t \mapsto (t, t^{-1})$

Infinitesimal character



Projection of the extended quotient on the ordinary quotient



Method for proving the local Langlands conjecture

$\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$

$$
\begin{array}{ccc}
 & T_\alpha // \Gamma_\alpha & \\
\swarrow & & \searrow \\
\widehat{G}_\alpha & & (\{\text{Langlands parameters}\}/{}^L G)_\alpha
\end{array}
$$

Method for proving the local Langlands conjecture

$\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$

$$\begin{array}{ccc} & T_\alpha//\Gamma_\alpha & \\ \swarrow & & \searrow \\ \widehat{G}_\alpha & & (\{\text{enhanced Langlands parameters}\}/{}^L G)_\alpha \end{array}$$

Theorem 1 (Aubert-Baum-Plymen-Solleveld)

Let G be a connected split reductive p-adic group, and let $\widehat{G}_\alpha$ be a Bernstein component of $\widehat{G}$ which is in the principal series of G. Then (granted a mild restriction on the residual characteristic of the p-adic field F over which G is defined) the ABPS conjecture is valid for $\widehat{G}_\alpha$.

$$\widehat{G}_\alpha \longleftrightarrow T_\alpha//\Gamma_\alpha$$

Theorem 1 (Aubert-Baum-Plymen-Solleveld)

Let G be a connected split reductive p-adic group, and let $\widehat{G}_\alpha$ be a Bernstein component of $\widehat{G}$ which is in the principal series of G. Then (granted a mild restriction on the residual characteristic of the p-adic field F over which G is defined) the ABPS conjecture is valid for $\widehat{G}_\alpha$.

$$\widehat{G}_\alpha \longleftrightarrow T_\alpha//\Gamma_\alpha$$

$\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$

$$\widehat{G}_\alpha \longleftarrow T_\alpha//\Gamma_\alpha \longrightarrow (\{\text{enhanced Langlands parameters}\}/{}^L G)_\alpha$$

Left arrow : representation theory of affine Hecke algebras.

Right arrow : Springer correspondence.

QUESTION. In the ABPS view of $\widehat{G}$, what are the L-packets?

CONJECTURAL ANSWER. Fix $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$. In the list $h_1, h_2, \ldots, h_r$ of correcting cocharacters (one h_j for each irreducible component of the affine variety $T_\alpha // \Gamma_\alpha$) there may be repetitions — i.e. it may happen that for $i \neq j$, $h_i = h_j$. It is these repetitions that give rise to L-packets.

Fix $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$. Let
$Z_1, Z_2, \ldots, Z_r$ be the irreducible components of the affine variety $T_\alpha // \Gamma_\alpha$.
Let $h_1, h_2, \ldots, h_r$ be the correcting cocharacters.
Let $\nu_\alpha : T_\alpha // \Gamma_\alpha \longrightarrow \widehat{G}_\alpha$ be the bijection of ABPS.
CONJECTURE. Two points $[(\gamma, t)]$, $[(\gamma', t')]$ have

$$\nu_\alpha[(\gamma, t)] \text{ and } \nu_\alpha[(\gamma', t')] \text{ are in the same L} - \text{packet}$$

if and only if

$$h_i = h_j \qquad \text{where } [(\gamma, t)] \in Z_i \text{ and } [(\gamma', t')] \in Z_j$$

and

$$c_i = c_j$$

and

$$\text{For all } \tau \in \mathbb{C}^\times, \;\; \theta_\tau[(\gamma, t)] = \theta_\tau[(\gamma', t')]$$

WARNING. An L-packet might have non-empty intersection with more than one Bernstein component. The conjecture does not address this issue. The statement of the ABPS conjecture begins

$$\text{Fix } \alpha \in \pi_o \text{Prim}(\mathcal{H}G).$$

So the ABPS conjecture assumes that a Bernstein component has been fixed — and then describes the intersections of L-packets with this Bernstein component.

Example

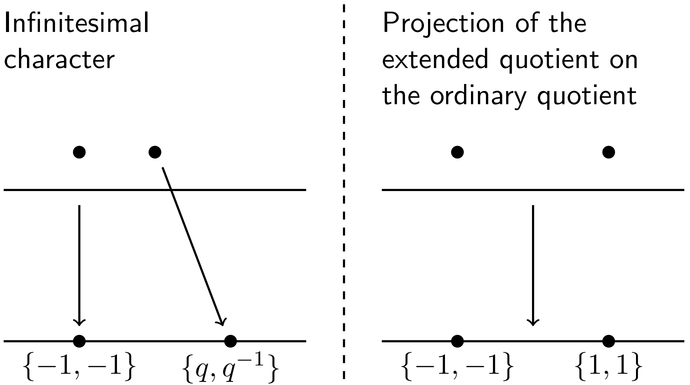
$G = SL(2, F)$
F can be any finite extension of the p-adic numbers $\mathbb{Q}_p$.
q denotes the order of the residue field of F.
$\widehat{G}_\alpha = \{$ Smooth irreducible representations of $GL(2, F)$ having a non-zero Iwahori fixed vector$\}$

$$T_\alpha = \{\text{unramified characters of the maximal torus of } SL(2, F)\} \\ = \mathbb{C}^\times$$

$$\Gamma_\alpha = \text{the Weyl group of } SL(2, F) = \mathbb{Z}/2\mathbb{Z}$$

$$0 \neq \gamma \in \mathbb{Z}/2\mathbb{Z} \qquad \gamma(\zeta) = \zeta^{-1} \qquad \zeta \in \mathbb{C}^\times$$

$$\mathbb{C}^\times//(\mathbb{Z}/2\mathbb{Z}) = \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) \bigsqcup \bullet \bigsqcup \bullet$$



Correcting cocharacter is $t \mapsto t^2$.

Preimage of $\{-1, -1\}$ is an L-packet.

Summary.

The extended quotient $T_\alpha//\Gamma_\alpha$, is (conjecturally) slightly non-canonically in bijection with the Bernstein component $\widehat{G}_\alpha$ and thus provides a setting in which precise book-keeping can be done for L-packets and correcting cocharacters.

Wiggly arrow indicates
"There is some interaction between the two conjectures."

Baum-Connes

$\updownarrow$ (wiggly)

ABPS

$\updownarrow$ (wiggly)

Local Langlands

Theorem (V. Lafforgue)

Baum-Connes is valid for any reductive p-adic group G.

Theorem (Harris and Taylor, G.Henniart)

Local Langlands is valid for GL(n, F).

Theorem (R. Plymen and J. Brodzki)

ABPS is valid for GL(n, F).

Representations of p-adic groups

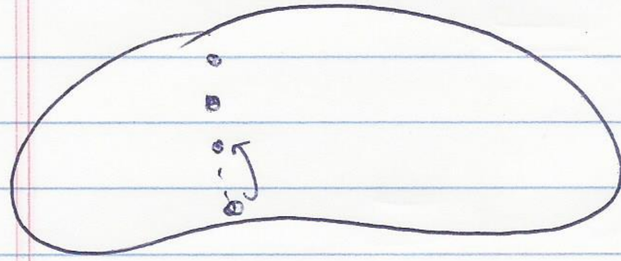
Paul Baum

Fri, Sept 5, 2014, 3:30–4:30pm

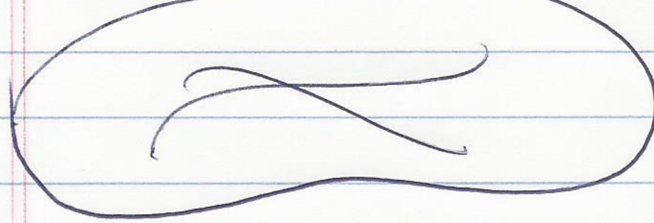
$G(\overline{F})$

$\widehat{G}$

Γ



X/Γ

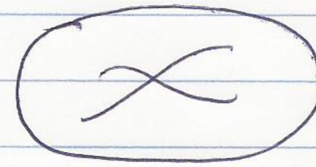


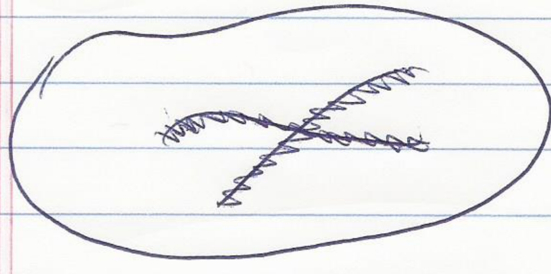
generically 1-1: away from a subvar, it is 1-1.

T_α / Γ_α,

$\Gamma_\alpha \subseteq W_G(M)$ Weyl.

$$\begin{array}{ccc} T_\alpha /\!/ \Gamma_\alpha & & \hat{G}_\alpha \\ \downarrow \rho_\alpha & & \downarrow \\ T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha \end{array}$$





$X \supseteq Y$

$\mathcal{O}(X) \oplus \mathcal{O}(Y)$

$X \sqcup Y$

$\mathcal{O}(X) \supseteq I_Y$

$$\begin{bmatrix} \mathcal{O}(X) & I_Y \\ I_Y & \mathcal{O}(X) \end{bmatrix}$$

$\downarrow$ $\mathbb{C} \oplus \mathbb{C}$

$\searrow$ $M_2(\mathbb{C})$ away from Y