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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Justin Hilburn **Email/Phone:** jhilburn@uoregon.edu

Speaker's Name: Joseph Bernstein

Talk Title: Periods and Global Invariants of Automorphic Representations

Date: 11 / 18 / 2014 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Period, Automorphic Representation, L-Function, Euler Product, Torsor

Please summarize the lecture in 5 or fewer sentences: Bernstein discussed global periods of automorphic representations and gave experimental evidence that they can be used to construct global invariants of such representations.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Joseph Bernstein - Periods and invariants of automorphic representations

Joint w Reznikov

K global field

K_p completion

$\mathbb{A} = \prod' K_p$

G algebraic group

G_p, $G = G_{\mathbb{A}}$, $G(K) = \Gamma$

Automorphic space $X = \Gamma \backslash G$ $\quad (\pi, G, C^\infty(X))$ representation

Automorphic representation

(π, G, V) - irreducible representation of adelic group

Automorphic structure $\nu : V \to C^\infty(X)$

$V = \otimes' V_p \qquad V \subseteq C^\infty(X)$

one strategy of assigning invariants to V is to construct invariants of V_p

we will consider a different strategy using periods. This is still in the experimental stage.

Period

$\mathcal{H} \subseteq \mathcal{G}$ $\quad H = \mathcal{H}(k)$ $\quad X_H = \Gamma_H / H$

$X_H \subseteq X$

$P_H : V \to \mathbb{C}$ $\quad v \mapsto \int_{X_H} v(\cdot)\big|_{X_H}\, d\mu$

$\chi : H \to \mathbb{C}^\times$ char trivial on Γ_H

$P_{H,\chi} : V \to \mathbb{C}_\chi$ $\quad v \mapsto \int \chi^{-1}(h)\, v(v)\, \chi(h)\, d\mu$

(H_1, χ_1) $\quad (H_2, \chi_2)$

$P_{H,\chi} \in P = \mathrm{Hom}_H(V, \mathbb{C}_\chi)$ period space

Consider Gelfand condition. For every place p
$((V_p)^*)^{(H_p, \chi_p)}$ is ~~...~~ ≤ 1 dimensional.

Main Construction Given (H_1, χ_1), (H_2, χ_2)
construct

$$I : P(\pi, \chi_1) \to P(\pi, \chi_2)$$

$I(P_{H_1}, \chi_1) / P_{H_2}\chi_2$ is an invariant of (π, V).

$\xi \in P(\pi, \chi_1) \subseteq V^*$

Informally: $I(\xi) = \int_{H_2} \chi_2^{-1}(h)\, \pi^*(h)\, \xi\, dh$ Need to make sense of this.

$$P(\pi, X_1) = \otimes' P(\pi_p, X_{1,p})$$
$$P(\pi, X_2) = \otimes' P(\pi_p, X_{2,p})$$

Procedure 1 : $\qquad I_p : P(\pi_p, X_{1,p}) \to P(\pi_p, X_{2,p})$

Procedure 2 : $\qquad I = \otimes' I_p$

tough $\qquad\qquad$ relatively straightforward

Def A torsor L is a 1d space over $\mathbb{C}$

Given a torsor L can define measures with values in L.
Get $I : F(X) \to L$. usual theory if $L = \mathbb{C}$.

G locally compact group, $S(G)$ smooth functions with compact support, $S(G)_G = S(G)/\langle g\varphi - \varphi \rangle \doteq L(G)$

torsor

$I : S(G) \to L(G)$

Mo -measure with values in $L(G)$.

Extends to $I : L^1(G) \to L(G)$

1: $L(G(\mathbb{A})) = \otimes' L(G(k_p))$

2: $X = \Gamma \backslash G$

$$I : S(X) \to L(G)$$
$$F(X) \to L(G)$$

Recall: Restricted $\otimes$.

$\{V_p\}$, p places

adelic structure = choice of $e_p \in V_p$ for almost all p.

$$V = \otimes' V_p$$

χ character on G $\quad I : S(G) \to L(G, \chi) \cong L(G) \otimes \mathbb{C}_\chi$

$$P(V, \chi) = \mathrm{Hom}_H(V, \mathbb{C}_\chi) \otimes L(H)$$

$$P_{H,\chi}(v) = \int_{X_H} \varphi(v)\big|_{X_H} \chi^{-1}(h)$$

Now to describe procedure 1

(H_1, χ_1) $\quad (H_2, \chi_2)$

$$I : P(\pi, \chi_1) \to P(\pi, \chi_2)$$
$$\otimes' P(\pi_p, \chi_{1,p}) \qquad \otimes' P(\pi_p, \chi_{2,p})$$

need auxiliary data of a top form δ on H_1

$$I_p : P(\pi_p, \chi_{1,p}) \to P(\pi_p, \chi_{2,p})$$

integral understood in weak sense

$$\xi : V_p \to L(H_p)$$

$$\xi' := \xi / |\delta| \in V_p^+ \qquad |\delta| \in L(H)$$

$$I_p(\xi) = \int_{H_2} \pi^*(\xi')\, \chi_2^{-1}(h)$$

$$I_p(\xi)(v) = \int_{H_2} (\pi(h)\xi')(v)\, \chi_2(h)$$

Procedure 2

$\xi \in P(\pi, \chi) = \otimes' P_p(\pi_p, \chi_p)$

$\xi = \prod \xi_p$

$I(\xi) = \prod I_p(\xi_p)$ for almost all p

$\xi_p = \xi_{p_0}$

$d_p = I_p(\xi_{p_0})$

$G = GL(2)$

$H_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $\chi_1 = \chi$

$H_2 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $\chi_2 =$ Hecke character

Existence of integrals equivalent to being able to continue some L-functions

by unfolding method

$I : P(\pi, \chi_1) \to P(\chi_2)$

$P_{H_1, \chi_1} \longmapsto P_{H_2, \chi_2}$

so global invariant is just 1

Now consider inverse

$H_1 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $\chi =$ Hecke

$H_2 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $\chi_1 = \chi$

using local methods

$I : P_{H_2, \chi_2} \to P_{H_1, \chi_1}$.

$P_{H_2, \chi_2} \longmapsto P_{H_1, \chi_1}$

global invariant is 1 again

$H_1 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $\quad X_1 = X$

$H_2 = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, $\quad X_2 = \psi'$

$I_p : P_{X_1, p} \to P_{X_2, p}$

$1 - q^{-1} \operatorname{tr} Ad(t_p)$

$\Pi_p \quad t_p \to d_p = \cancel{1 - q^{-1} \operatorname{tr} t_p}$

$\nearrow$ ~~$\mathbb{Z}$~~

Partial L-functions

$L_p = \det(1 - q^{-1} \operatorname{tr} Ad(t_p))$

$\prod d_p / L_p$ is absolutely convergent

$\prod d_p := \prod \frac{d_p}{L_p} \, L(\operatorname{tr} ad, 1)$

In summary: 1) These invariants are given by integrals and hence can be estimated.

2) Waldspurger

$G = PGL(2) \quad T \subset G$ torus

$G \times G \qquad \Delta = H_1 \quad \chi = 1$

$T \times T = H_2 \quad \chi \times \chi^{-1}$

$\pi \otimes \hat{\pi}$

Waldspurger showed

$P_{H_2, \chi} \mapsto P_{H_2, \chi_2}$

and hence global invariant is 1.