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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Justin Hilburn **Email/Phone:** jhilburn@uoregon.edu

Speaker's Name: Peter Scholze

Talk Title: Some structures from Geometric Langlands in the p-adic setting

Date: 11 / 17 / 2014 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Beilnson-Drinfeld Grassmannian, p-adic field, Witt vectors, Perfectoid algebras, diamonds,

Please summarize the lecture in 5 or fewer sentences:
Using the theory of perfectoid spaces, Peter describes the analogues of the Affine and Beilinson Drinfeld Grassmannians in the setting of a p-adic field. The special fiber is a Witt Grassmannian and the generic fiber is a space similar to one constructed by Fontaine.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
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Peter Scholze - Some structures from geometric Langlands in the setting of a p-adic field

Spec $\mathbb{Z}$

?

C/k smooth projective curve

have affine Grassmannians

or even Beilinson-Drinfeld

Grassmannian $Gr^{BD}_{C^n} \to C^n$

For simplicity assume, $G = GL_n$

Recall: $Gr^{BD}_{C^n} \longrightarrow C^n$ is a functor over C^n

whose S-points are:

$$Gr^{BD}_{C^n}(S) = \left\{ (\xi, x_1, \dots, x_n) \;\middle|\; \begin{array}{l} \xi \;/\; C \times_k S \text{ vector bundle of rank } k,\ x_i \in C(S), \\ \xi|_{C\times S - \cup \Gamma_{x_i}} \cong \mathcal{O}^k_{C\times S - \cup\Gamma_{x_i}} \end{array} \right\} \Big/ \sim$$

In particular the fiber over $(x_1, \dots, x_n) \in C^n$ when the x_i are pairwise distinct is

$$\prod_{i=1}^{m} Gr_{x_i}$$

by Beauville-Laszlo.

completion of C at x_i

$$Gr_{x_i}(S) = \left\{ \begin{array}{l} \xi \text{ Vb of rank } k \text{ on } \hat{C}_{x_i} \times S \\ + \ \xi|_{(\hat{C}_{x_i} - \{x_i\}) \times S} \cong \mathcal{O}^r_{\hat{C}_{x_i} - \{x_i\}} \end{array} \right\} \Big/ \sim$$

Gr_{x_i} is called the ~~affine~~ Grassmannian at x_i.

Points colliding $\rightsquigarrow$ "Fusion product" in
Geometric ~~[illegible]~~
Satake

Want a similar picture for Spec $\mathbb{Z}$

Two issues

1) What is Spec $\mathbb{Z} \times$ Spec $\mathbb{Z}$?

occurs $n>1$ $\rightarrow$ In particular, how do points collide?

2) In 'moduli' problem what is

occurs $n=1$ $\rightarrow$ Spec $\mathbb{Z}$ $\times S$

Goal: Resolve this issues after completions at a prime p.

What is $Gr^{BD}_{\mathrm{Spec}\,\mathbb{Z}_p} \longrightarrow \mathrm{Spec}\,\mathbb{Z}_p$

Special fiber: With vector Grassmannian
= mixed char affine Grassmannian.

Should be a functor

R ring of char p $\longmapsto$ $\left\{ \begin{array}{l} \xi \text{ bundle of rk } k \\ \text{over "Spec } \mathbb{Z}_p \times \text{Spec } R\text{"} \\ + \ \xi\big|_{\text{"Spec } \mathbb{Q}_p \times \text{Spec } R\text{"}} \cong \mathcal{O}^r_{\text{"Spec } \mathbb{Q}_p \times \text{Spec } R\text{"}} \end{array} \right\}$

Principle If R is a perfect ring of char p

$$\left(\Phi : \begin{array}{c} R \longrightarrow R \\ x \longmapsto x^p \end{array} \quad \text{is an isomorphism} \right)$$

then

$$\text{"Spec } \mathbb{Z}_p \times \text{Spec } R\text{"} := \text{Spec } W(R)$$
$$\| $$
$$\left\{ \sum_{n=0}^{\infty} [a_n] p^n \;\middle|\; a_n \in R \right\}$$

If R not perfect $W(R)/p \not\cong R$

$$\text{"Spec } \mathbb{Q}_p \times \text{Spec } R\text{"} = \text{Spec } W(R)[\tfrac{1}{p}]$$

Remarks 1) There is a projection map

$\text{Spec } W(R) \longrightarrow \text{Spec } \mathbb{Z}_p$ with

special fiber $\text{Spec } R$

However there is no projection

$$\underset{\text{char } 0}{\text{Spec } W(R)} \longrightarrow \underset{\text{char } p}{\text{Spec } R}$$

2) On topological spaces underlying adic spaces there is a map

$$\text{adic spectrum} \hookrightarrow \text{Spa } W(R) \longrightarrow \text{Spec } R$$

Thus we should define

$$\begin{array}{c} R \text{ ring} \\ \text{char } p, \\ \text{perfect} \end{array} \longmapsto \left\{ \begin{array}{l} \{ \text{vb of rk } r \text{ over Spec } W(R) \} \\ + \{ \mathcal{L}_{\text{Spec } W(R)[\frac{1}{p}]} = \mathcal{O}^r_{\text{Spec } W(R)[\frac{1}{p}]} \} \end{array} \right\}$$

Thm (Zhu) The Functor Gr^{witt} is represented by an ind-(perfect algebraic space). In other words étale locally the strata of Gr^{witt} are perfect schemes.

Question Does Gr^{witt} have some kind of "finite type" structure?

e.g. miniscule Schubert cells

$$\left\{ pW(R)^r \subseteq \xi \subseteq W(R)^r \text{ st } W(R)^r/\xi \text{ rk } d \text{ over } R \right\}_{/\sim}$$

Same phenomena happens for ordinary Affine Grassmannian

are still ordinary Grassmannians over R.

Next, consider $Gr^{BP}_{\operatorname{Spec} \mathbb{Z}_p} \longrightarrow \operatorname{Spec} \mathbb{Z}_p$ or rather what is the generic fiber over $\operatorname{Spec} \mathbb{Q}_p$.

Here test objects should be $\mathbb{Z}_p$-algebras but we want some perfectness condition.

Unfortunately perfectness is determined by Frobenius and $\mathbb{Z}_p$ has char 0.

Fact 1) There is a notion of perfectoid $\mathbb{Z}_p$-algebra
(e.g. $\mathbb{C}_p \langle T^{1/p^\infty} \rangle$
& these are Banach algebras)

2) If R is a perfectoid $\mathbb{Z}_p$-algebra
there is a ring $B^+_{dR}(R) \overset{\theta}{\twoheadrightarrow} R$,
complete with respect to $\ker(\theta)$-adic topology

$$\ker(\theta) = (\xi) \leftarrow \text{non zero divisor}$$

$$\xi^i B^+_{dR}(R) / \xi^{i+1} B^+_{dR}(R) \simeq R$$
$\leftarrow$ noncanonically

If R has char p, then
$$B^+_{dR}(R) = W(R) \qquad \xi = p$$

$B^+_{dR}(\mathbb{C}_p)$ = Fontaine's B^+_{dR}
a complete DVR with residue field
$\mathbb{C}_p$ so $B^+_{dR}(\mathbb{C}_p) = \mathbb{C}_p[[\xi]]$
$\leftarrow$ non canonical

Idea: Spf $B^+_{dR}(R)$ = completion of
"Spec $\mathbb{Z}_p \times$ Spec R" along
Γ: Spec $R \to$ "Spec $\mathbb{Z}_p \times$ Spec R"

<u>Def</u> $Gr^{B_{dR}^+}$ is the functor

$$\{\text{perfectoid } \mathbb{Q}_p\text{-alg}\} \longrightarrow \text{Sets}$$

$$R \longmapsto \left\{ \begin{array}{ll} \Xi \subset B_{dR}(R)^r & \text{fin proj} \\ B_{dR}^+(R)\text{-module} & \text{of rank } R \\ \Xi[\tfrac{1}{\xi}] = B_{dR}(R)^r & . \end{array} \right\} \Big/ \sim$$

<u>Thm (S)</u> This defines an ind-"diamond"

(i.e., it is pro-étale locally representable by perfectoid $\mathbb{Q}_p$-alg)

<u>e.g.</u> $\mathbb{G}_m / \mathbb{K}_p$ $= \mathbb{Q}_p[T^{\pm}]$ is not representable by a perfectoid alg but $\mathbb{C}_p[T^{\pm}, {}^{1/p^\infty}]$ is perfectoid.

<u>Remark</u> Formalism is not optimal. It is not easy to say that $Gr^{BD}_{\operatorname{Spec}\mathbb{Z}_p} \to \operatorname{Spec}\mathbb{Z}_p$ exists with special fiber Gr^{Witt} and generic fiber $Gr^{B_{dR}^+}$.

Next $Gr^{DD}_{(\operatorname{Spec}\mathbb{Z}_p)^2} \to \operatorname{Spec}\mathbb{Z}_p \times \operatorname{Spec}\mathbb{Z}_p$

test objects: alg R with two structure maps

$$\mathbb{Z}_p \to R$$

makes no sense!!

<u>Fact</u> There is a "tilting" functor

$$\{\text{perfectoid } \mathbb{Z}_p\text{-alg}\} \longrightarrow \{\text{perfect } \underset{\text{Banach}}{\wedge} \mathbb{F}_p\text{-algebras}\}$$

$$R \longmapsto R^b = \varprojlim_{x \mapsto x^p} R$$

$$(\operatorname{Spa} R \simeq \operatorname{Spa} R^b)$$

$$\begin{array}{l} \mathbb{C}_p \rightsquigarrow \overline{\mathbb{F}_p((t^{\circ}))} \\ \| \\ \widehat{\overline{\mathbb{Q}}_p} \end{array}$$

$$\mathbb{C}_p\langle T^{1/p^\infty}\rangle \rightsquigarrow \widetilde{\overline{\mathbb{F}_p((t))}}\langle T^{\pm 1/p^\infty}\rangle$$

One can make sense of idea

$$\{\text{perfectoid } \mathbb{Z}_p\text{-alg}\} \simeq \left\{\begin{array}{l}\text{perfectoid } \mathbb{F}_p\text{-alg} \\ + \text{"structure map"} \\ \text{to } \operatorname{Spec} \mathbb{Z}_p\end{array}\right\}$$

$$R \longmapsto (R^b, \operatorname{Spd} R^b \to \operatorname{Spd} \mathbb{Z}_p)$$

Thus test objects

$$= \left\{\begin{array}{l}\text{perfectoid } \mathbb{F}_p\text{-al } R + \text{two structure} \\ \text{maps } \operatorname{Spd} R \to \operatorname{Spd} \mathbb{Z}_p\end{array}\right\}$$

$$\simeq \{S_1, S_2 \text{ perfectoid } \mathbb{Z}_p\text{-alg} + \text{iso } S_1^b \simeq S_2^b\}$$

Thm (S) In freq setup one can define

BD-Grassmannian $Gr^{BD} \to (\mathrm{Spec}\, \mathbb{Z}_p)^n - \{\text{closed point}\}$

as ind-diamond

over closed point, should still get Gr^{Witt}.

Again it is unclear how to relate special fiber and generic fiber.

To get Bun_G need global story.
Perverse sheaves should be possible