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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Justin Hilburn **Email/Phone:** jhilburn@uoregon.edu

Speaker's Name: Helene Esnault

Talk Title: Relation between the etale fundamental group and stratifications

Date: 11 / 19 / 2014 **Time:** 9 : 30 **am / pm (circle one)**

List 6-12 key words for the talk: Etale fundamental group, Tannakian fundamental group, crystals, iso-crystals, d-modules

Please summarize the lecture in 5 or fewer sentences:

Esnault surveyed various generalizations of the Grothendieck-Malcev theorem which relates the etale fundamental group to the Tannakian fundamental group of the category of local systems. In positive characteristic there is no analogue of the topological fundamental group so things are much more difficult.

CHECK LIST

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☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
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Hélène Esnault – Relation between the Etale fundamental group and stratifications

$/\mathbb{C}$ smooth variety

(finite local systems) $\subseteq$ (local systems) ← Tannakian fundamental gp.

$$\begin{tikzcd} \pi_1^{et}(X) & \pi^{loc\,sys}(X) \arrow[l, two heads] \arrow[r, hook] & ? \end{tikzcd}$$

$\parallel$ Riemann existence thm

$\pi_1^{loc\,sys}(X)^{\wedge f}$ profinite completion

Q! what information is in kernel?

Theorem (Malčev – Grothendieck) If the etale fundamental group $\pi_1^{et}(X) = \{1\}$ then $\pi^{loc\,sys}(X) = \{1\}$

Pf

$$\begin{tikzcd} \pi_1^{top}(X(\mathbb{C})) \arrow[r, "\wedge f"] \arrow[dr, "\wedge alg"'] & \pi^{et}(X) \\ & \pi^{loc\,sys}(X) \end{tikzcd}$$

group of finite type

So any rep $\rho: \pi_1^{top}(X) \to GL(r, \mathbb{C})$ factors

$$\begin{tikzcd} \pi_1^{top}(X) \arrow[r] \arrow[dr] & GL(r,\mathbb{C}) \\ & GL(r, A) \arrow[u] \end{tikzcd}$$

finite type

$\rho \neq 1 \Leftarrow \rho_{k(a)} \neq 1 \qquad a \xrightarrow{\text{closed pt}} |A|$ $\square$

70's) Mumford – Gieseker – Katz

Char $p > 0$ analogs

$/\mathbb{C}$ (local systems) $\overset{\text{R.H.}}{\longleftrightarrow}$ reg. singular $\mathcal{O}$-coherent D-modules

$\longleftrightarrow$ regular singular crystals in the infinitesimal site.

$\longleftrightarrow$ regular singular alg flat connections

Should assume X smooth

$k = \bar{k}$ char $k > 0$ connections are $\mathcal{O}_X^{(\infty)}$-linear.

2 ways to go:

$\mathcal{O}$-coh D-modules (v.s.) k-linear & tannakian

$\updownarrow$

Frobenius divided sheaves $\Leftrightarrow$ $\mathcal{O}$-coh crystals on infinitesimal site

$\pi^{\infty\text{crys}} \twoheadrightarrow \pi^{\text{Et}}(X) = (\pi^{\infty\text{crys}})^{\wedge f}$

dos Santos in setting of D-modules

K-linear where $K = \mathrm{Frac}(W(k))$ with vectors

$\downarrow$

(X projective) $\mathcal{O}$-coh crystal in crystalline site.

iso-

$\pi^{\text{isocryst}} \twoheadrightarrow \pi^{\text{ét}}(X) \otimes_K \bar{K}$

Conjecture X projective

1) $\pi^{\text{et}}(X) = 1 \Rightarrow \pi_1^{\infty\text{crys}}(X) = 1$? (Gieseker)

2) $\pi_1^{\text{et}}(X) = 1 \Rightarrow \pi^{\text{isocrys}}(X) = \{1\}$? (de Jong)

L. Kindler

X quasi-projective (K. Schmidt)

reg sing $\mathcal{O}$-coh $\mathcal{D}$-modules (no assumption of res of sing)

$$\begin{tikzcd} \pi^{\infty\text{-crys}} \arrow[r, two heads] \arrow[d] & \pi_1^{\text{ét}} \arrow[d] \\ (\pi^{\text{lo},\text{crys}})^{\wedge p} \arrow[r] & \pi_1^{\text{ét},\text{tame}} \end{tikzcd}$$

refinement: ~~...~~ X not proper

If $\pi_1^{\text{ét}} = 1$, $\pi^{\infty\text{-crys}} = 1$

If $\pi_1^{\text{ét},\text{tame}} = 1$, $\pi^{\text{reg sing crys}} = 1$

Q: what is analogue of topological fundamental group in char p.

Thm X projective

$$\pi_1^{\text{ét}} = 1 \Rightarrow \pi^{\infty\text{-crys}} = \{1\}$$

=

X not proper and for $\begin{cases} \pi_1^{\text{ét},t} & \text{no result.} \\ \pi_1^{\text{ét}} & \text{some results in } \mathbb{F}_p \end{cases}$

($\mathbb{A}^n$ ok)

analogy for π_1^{top} being of f.t.

X ~~...~~

$$\mathcal{E} = (E_n, E_{n+1}, \ldots, \sigma_n) \quad \text{v.b.}$$

Frobenius

$$\sigma_n : F^* E_{n+1} \underset{\text{iso}}{\simeq} E_n$$

$$\mathcal{E}(n) = (E_n, E_{n+1}, \ldots, \sigma_n, \ldots)$$ ~~...~~

extension of ∞-crystals ~~slope stable~~

$n \gg 0$ all E_n ~~of Hilbert polyn~~.

slope stable of Hilbert ~~...~~ polynomial $P_{\mathcal{O}}$.

- Langer. existence of quasi-projective moduli spaces of $\mathcal{H}$-stable V.B. w/ fixed P_Θ.
- Specialization of coefficients of monodromy $\Rightarrow$ $/\mathbb{F}_p$ $\Leftrightarrow$ Hrushovsky (Varshavsky) existence of closed periodic points.

X non-proper smooth

$X \overset{j}{\hookrightarrow} \bar{X}$ normal open embedding

Deligne (relying on π_1^{top} of f.t.)

(E, ∇) r.s. extend to $(\bar{E}, \bar{\nabla})$ on $\bar{X}$

$\{c_i(\bar{E})\} \in H^{2*}(\bar{X})$

finitely set

Analogue to Deligne

$\exists$ finitely many

$\mathcal{X}_1(n) \in \mathbb{Q}[n]$

st $\forall$ $(E_0, E_1, \ldots)$ on X there exists n_0 such that for all $n > n_0$

$\mathcal{X}(j_* E_m(n+1))$

$\mathbb{Q}[m]$

is one of the $\mathcal{X}_1$

Isocrystals, $k = \bar{k}$ char $p > 0$

$$Y \xrightarrow{f} X$$
$$\rightsquigarrow$$
$$X_W \xrightarrow{f_W} Y_W$$

$$\underbrace{R^1 f_{W,*}\, \Omega_{Y_W/X_W}}$$

$f_W \Omega^1_* \mathcal{U}/X_W$ positive

as isocrystal "often" stable, but as sheaves ~~...~~ unstable by work of Griffiths.

<u>prop</u> de Jong's conjecture is true for rank 1 isocrystals $(p \geq 3)$

- $H^1_{crys}(\mathcal{O}^X_{X/W}) = \{1\}$
- $Ext^1_{crys}(H, H) = 0$

Proof reduced to case of irreducible isocrystals.

<u>Theorem (Shiho)</u> X smooth projective $/k = \bar{k}$ assume $\pi_1^{ét}(X) = 1$. Then any isocrystal $\mathcal{E}$ st $\exists\, E$ crystal with $E \otimes \mathbb{Q} = \mathcal{E}$, st $E(X)$ torsion free and ~~...~~ strongly semi-stable of Hilbert polynomial P_Θ.

2 Ingredients

- Jum (E, Mehta) $\Rightarrow$ the only strongly semistable torsion free coherent sheaves with Hilbert polynomials $P_{\mathcal{O}}$ are the trivial ones $\bigoplus^{r} \mathcal{O}$.

- Berthelot: level increasing Frobenius pullback of isocrystals.